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WEIGHTED NORM INEQUALITIES FOR STRONGLY SINGULAR CONVOLUTION OPERATORS

BY
SAGUN CHANILLO

ABSTRACT. We derive sharp function estimates for convolution operators whose kernels are more singular than Calderón-Zygmund kernels. This leads to weighted norm inequalities. Weighted weak (1,1) results are also proved. All the results obtained are in the context of A_p weights.

0. Introduction. We begin by recalling notations relevant to R^n. Given a function $f(x)$, its Fourier transform will be denoted by

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{-i\langle x,\xi\rangle}\,dx.$$

Throughout this paper $v(x)$ and $w(x)$ will denote nonnegative locally integrable functions. We let

$$\|f\|_{p,w} = \left(\int_{R^n} |f|^p w\,dx\right)^{1/p}, \qquad 0 < p < \infty.$$

When $w \equiv 1$, we simply write $\|f\|_p$. Given $E \subset R^n$. We let $w(E) = \int_E w\,dx$ and $|E|$ will, as usual, stand for the Lebesgue measure of E. Given a cube $I \subset R^n$, cI, $c > 0$, will denote a cube concentric with I but with diameter c times that of I.

Let $\theta(\xi)$ be a smooth radial cut-off function $\theta(\xi) = 1$ if $|\xi| \geqslant 1$ and $\theta(\xi) = 0$ if $|\xi| \leqslant 1/2$. In this paper we wish to study the multipliers

$$\widehat{T_{b,a}f}(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^a}\hat{f}(\xi),$$

where $0 < b < 1$ and $0 < a \leqslant nb/2$. When $a = nb/2$ the resulting operator will be denoted by $T_b f(x)$. These operators have been the subject of much study. In the periodic situation they were investigated by Hardy-Littlewood and A. Zygmund [**18**], and also I. I. Hirschman [**8**]. The study of these operators in the context of L^p spaces was carried out by Hirschman [**8**], and S. Wainger [**17**]. Sharp endpoint estimates were obtained by C. Fefferman and E. M. Stein in [7] using the duality of H^1 and BMO. To be more precise, one has

$$(0.1) \qquad \|T_b f\|_p \leqslant c_p \|f\|_p, \qquad 1 < p < \infty.$$

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(0.2) If $0 \leqslant a < nb/2$, then for p, such that $|1/p - 1/2| \leqslant a/nb$, we have,

$$\|T_{b,a}f\|_p \leqslant c_p\|f\|_p.$$

Moreover,

$$|\{x: |T_b f(x)| > \lambda\}| \leqslant \frac{c}{\lambda}\|f\|_1, \qquad \lambda > 0. \tag{0.3}$$

The inequality (0.3) is due to C. Fefferman [**6**]. The p ranges in (0.1) and (0.2) are best possible as has been shown by Wainger [**17**]. The operators $T_{b,a}$ are not Hörmander-Mihlin multipliers but furnish examples of multipliers with $S_{\rho,\delta}^{-m}$ symbols, with $m \geqslant 0$, $\rho < 1$. In this context these multipliers were studied by L. Hörmander [**9**].

Our aim is to examine conditions on $w(x)$ so that the estimate $\|T_{b,a}f\|_{p,w} \leqslant c_p\|f\|_{p,w}$ holds. This question has been the subject of much study by several authors in the case when the multipliers are Hörmander multipliers or Calderón-Zygmund kernels like, say, the Hilbert transform on the line. For example, in [**10, 2**] it was shown that a necessary and sufficient condition on $w(x)$, for the validity of the estimate $\|Hf\|_{p,w} \leqslant c_p\|f\|_{p,w}$, $1 < p < \infty$ (H = Hilbert transform), is

$$\sup_I \left(\frac{1}{|I|}\int_I w\,dx\right)\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}dx\right)^{p-1} \leqslant c. \tag{0.4}$$

Here I denotes an interval in R; in R^n, I is taken to be a cube. (0.4) is referred to as the A_p condition. The techniques of estimating Calderón-Zygmund kernels so as to derive weighted estimates are not adequate in the case of the strongly singular operators T_b. The reason is rather simple. The kernel for T_b is very singular. Roughly speaking, it looks like $K_{b'}(x) = e^{i|x|^{-b'}}/|x|^n$, $b' = b/(1-b)$. Indeed the cancellation is minimal and if one makes a quick computation for $|x| \geqslant 2|y|$, one gets

$$|K_{b'}(x-y) - K_{b'}(x)| \leqslant c|y|/|x|^{n+b'+1}.$$

Thus, in comparison with the term $|y|/|x|^{n+1}$ obtained from Calderón-Zygmund kernels, we expect local problems. The point of view we adopt to estimate the strongly singular kernels is through a function called the sharp function introduced by Fefferman and Stein in [**7**]. This is given by

$$f^{\#}(x) = \sup_{x \in I} \frac{1}{|I|}\int_I |f - f_I|\,dx,$$

where I is a cube in R^n and $f_I = |I|^{-1}\int_I f\,dx$. The use of the sharp function for weight problems in the context of Calderón-Zygmund kernels has been made before by D. Kurtz and R. L. Wheeden in [**11**]. The motivation behind this comes from Theorem 1 in [**7**]. It is proved there that the operator T_b maps bounded functions into BMO. BMO is the class of locally integrable functions such that $\|f^{\#}\|_\infty < \infty$. However, the proof in [**7**] is L^2 in nature and will not yield sharp results in the context of A_p weights. For example, one fact that one could get is that, for $2 < p < \infty$ and $w \in A_{p/2}$, $\|T_b f\|_{p,w} \leqslant c_p\|f\|_{p,w}$. The way to get sharper results is to make more effective use of the minimal cancellation present in the kernel for T_b. This is the content of Lemma (2.1). This enables us to show the following

THEOREM A. *Let* $w \in A_p$, $1 < p < \infty$. *Then*

(0.5) $$\|T_b f\|_{p,w} \leqslant c_p \|f\|_{p,w}.$$

Theorem A and interpolation allows us to prove the next result.

THEOREM B. *Let* $\alpha = nb|1/p - 1/2|$, *and* $w \in A_p$. *Then for* $1 < p < \infty$, $\alpha \leqslant a \leqslant nb/2$, *and for* γ, *such that* $\gamma = (a - \alpha)/(nb/2 - \alpha)$, *we have*

$$\|T_{b,a} f\|_{p,w^\gamma} \leqslant c_p \|f\|_{p,w^\gamma}.$$

We now consider the case $p = 1$. We say $w \in A_1$ if $|I|^{-1}\int_I w\,dx \leqslant c \operatorname{ess\,inf}_I w$. The main result then is as follows.

THEOREM C. *Let* $w \in A_1$. *Then for* $\lambda > 0$

$$w\{x: |T_b f(x)| > \lambda\} \leqslant \frac{c}{\lambda}\|f\|_{1,w}.$$

The proof of Theorem C is through an analysis of the multiplier, while for Theorem A it is through the kernel. As pointed out for $w \equiv 1$, Theorem C is due to Fefferman [**6**]. The proof of (0.3) is L^2 in nature. For weight functions in A_1 this is not adequate and an L^p proof is needed for $1 < p < 2$. The reason is that Plancherel's formula fails for A_1 weights. This is done here by using the theory of weighted H^p spaces. We say a tempered distribution f is in H^p_w, $0 < p < \infty$, if $\|M(f)\|_{p,w} < \infty$. $M(f)(x)$ is the maximal function defined by

$$M(f)(x) = \sup_{|x-y|<t} |f * \phi_t(y)|.$$

Here $\phi \in C_0^\infty(R^n)$ with $\int_{R^n}\phi(x)\,dx = 1$ and $\phi_t(x) = t^{-n}\phi(x/t)$. The spaces H^p_w have been investigated in detail by J.-O. Strömberg and A. Torchinsky [**14–16**]. However, just by making up an L^p proof of (0.3) the proof of Theorem C does not follow. There is a deeper objection. In the proof of Theorem C, it is necessary to localize the problem onto selected intervals. At this stage one uses estimates related to fractional integrals. The presence of the weight prevents us from employing these estimates right away. One way, one would hope as in the case of Calderón-Zygmund kernels, is to use the so-called reverse Hölder inequality; but this worsens the estimates. The idea then is to effectively localize the weight in a manner which is compatible with the Calderón-Zygmund decomposition of f. This is Lemma (3.9). This lemma has also proved helpful in other situations, with other types of oscillating kernels. This will be discussed in a forthcoming paper on oscillating kernels with D. Kurtz and G. Sampson.

We would like to point out that the condition A_p is not necessary for Theorem A to hold. For example, if $w(x) = (1 + |x|)^\alpha$, $\alpha \in R$, then for any p, $1 < p < \infty$, (0.5) does hold. This is very easy to see and is shown in §5. In the context of A_p weights, however, Theorem A is sharp as the next result shows.

THEOREM D. *Let* $w(x)=|x|^{\alpha}$, $\alpha \leqslant -n$ *or* $\alpha \geqslant n(p-1)$. *Then for* $1<p<\infty$, $\|T_b f\|_{p,w} \leqslant c_p\|f\|_{p,w}$ *fails.*

This paper is organized as follows. §1 contains some basic results for A_p weights and maximal functions. §2 contains the proof of Theorem A. In §3 we collect facts about weighted Hardy spaces H_w^p and develop the machinery needed to obtain a proof of Theorem C. §4 contains the proof of Theorem C, and the last section, §5, contains the proofs of Theorems B and D. To maintain the presentation short and clear we present complete proofs of all the results only when $n=1$. The extension to n-dimensions is discussed in remarks following the proofs of the lemmas. The extension to R^n in most cases is evident. The R^n version of Lemma (2.1) may be deduced by using Hankel transform formulas of Chapter 4 of [**13**]. The basic outline of the proof in R^n still follows the $n=1$ case. There are, however, more error terms to handle arising from the asymptotics of Bessel functions and the resulting computation is tedious and unilluminating and not central to the main developments. A partial example of what one may encounter in such a computation is in the remarks following Theorem D, where the n-dimensional proof of Theorem D is sketched. The one exception to this is the main Lemma (3.9) which is geometric and a proof has been provided for R^n. The symbol c will as usual denote a generic constant dependent on the dimension and possibly on the parameters b, a of the operator $T_{b,a}$. $\hat{C}_{0,0}^{\infty}$ will be the dense subspace of H_w^p of smooth functions such that $\hat{f} \in C_0^{\infty}(R^n)$ and such that 0 does not lie in the support of $\hat{f}$; see [**16**].

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1. Preliminary lemmas.

THEOREM (1.1). *The kernel for the multiplier operator* $T_b f(x)$ *is given by*

$$c\frac{e^{i\alpha_b|x|^{-b'}}}{|x|^n}\chi(|x| \leqslant 1)+h(x), \qquad b'=b/(1-b),$$

with $|h(x)| \leqslant c(1+|x|)^{-(n+1)}+c|x|^{-n+\varepsilon}\chi(|x| \leqslant 1)$, $\varepsilon>0$. *Here* $\alpha_b=b^{b/(1-b)}-b^{1/(1-b)}$ *and* ε *depends only on* b.

Theorem (1.1) is due to Wainger [**17**, p. 41].

We define the Hardy-Littlewood maximal function as follows:

$$(|f|^r)^{*1/r}(x)=\sup_{x \in I}\left(\frac{1}{|I|}\int_I|f|^r dx\right)^{1/r}, \qquad 1 \leqslant r<\infty,$$

I a cube in R^n.

The following lemma follows from Theorem (1.1) by a standard argument.

LEMMA (1.2). *Let*

$$K_{b'}(x)=\frac{e^{i\alpha_b|x|^{-b'}}}{|x|^n}\chi(|x| \leqslant 1),$$

$$b'=b/(1-b) \quad \text{and} \quad \alpha_b=b^{b/(1-b)}-b^{1/(1-b)}.$$

Then

$$|T_b f(x)| \leqslant c(|K_{b'} * f(x)| + f^*(x)).$$

Moreover, in the inequality above the roles of $T_b f(x)$ and $K_{b'} * f(x)$ may be interchanged.

From the lemma above and the L^p results for $T_b f(x)$ it follows by changing variables, that for any $\alpha_b \neq 0$ and real if we let

$$K_b(x) = \frac{e^{i\alpha_b|x|^{-b}}}{|x|^n}\chi(|x| \leqslant 1), \qquad 0 < b < \infty,$$

then

(1.3) $$\|K_b * f\|_p \leqslant c_p\|f\|_p, \qquad 1 < p < \infty.$$

THEOREM (1.4). *Let $w \in A_p$. Then there exists an r_0, $1 < r_0 \leqslant p$, such that for all r, $1 \leqslant r < r_0$,*

(1) $$\|(|f|^r)^{*1/r}\|_{p,w} \leqslant c_p\|f\|_{p,w}, \qquad 1 < p < \infty.$$

Let $w \in A_1$. Then

(2) $$w\{x: f^*(x) > \lambda\} \leqslant (c/\lambda)\|f\|_{1,w}.$$

The result above is basic and due to B. Muckenhoupt. A proof may be found in **[2]**. The next result we need is due to A. Cordoba and Fefferman **[5]**. To write it down, recall that we say $w \in A_\infty$ if $w \in A_p$, for some p, $1 \leqslant p < \infty$.

THEOREM (1.5). *Let $w \in A_\infty$ and $\|f^*\|_{p,w} < \infty$. Then for $0 < p < \infty$,*

$$\|f^*\|_{p,w} \leqslant c_p\|f^\#\|_{p,w}.$$

For a further discussion of A_p weights we refer to **[2]**. However, in the sequel the following facts prove useful. The proofs of the statements below may be found in **[2]**.

THEOREM (1.6). *Let $w \in A_\infty$, then*:

(1) *There exists an r, $1 < r < \infty$, such that for any cube $I \subset R^n$,*

$$\left(\frac{1}{|I|}\int_I w^r\,dx\right)^{1/r} \leqslant \frac{c}{|I|}\int_I w\,dx.$$

(2) *Let $B(x_0,r) = \{x: |x - x_0| < r\}$. Then, for $t \geqslant 1$,*

$$w(B(x_0,tr)) \leqslant ct^{nd}w(B(x_0,r)),$$

for some $d \geqslant 1$. We will refer to this fact by saying that w is doubling.

(3) *If $w \in A_p$, then $w \in A_q$, for $q \geqslant p$.*

2. The L_w^p inequalities for $1 < p < \infty$. For $x \in R$, let

$$\tilde{K}_{b,p}(x) = \frac{e^{i|x|^{-b}}}{|x|^{(2+b)/p}}, \qquad 0 < b < \infty, 2 \leqslant p \leqslant \infty.$$

LEMMA (2.1). *Let* $(2+b)/p<1$. *Then*,

$$\left\|\tilde{K}_{b,p}*f\right\|_p \leqslant c_p\|f\|_{p'}, \qquad 1/p+1/p'=1.$$

PROOF. We shall prove the lemma by using Stein's theorem on complex interpolation of operators [**13**]. We now define, for $z=u+iy$, $0\leqslant u\leqslant 1/2$,

$$\tilde{K}^{\varepsilon}_{b,z}(x)=\begin{cases}\dfrac{e^{i|x|^{-b}}}{|x|^{(2+b)z}}, & \text{if } |x|>\varepsilon,\\ 0, & \text{if } |x|\leqslant\varepsilon.\end{cases}$$

We then define the operator $A^{\varepsilon}_{b,z}$ by $A^{\varepsilon}_{b,z}f(x)=\tilde{K}^{\varepsilon}_{b,z}*f(x)$. It is easily seen that for fixed $\varepsilon>0$, $A^{\varepsilon}_{b,z}$ is an analytic family of operators in the sense of [**13**, p. 205]. It may be seen from the definitions that if $\operatorname{Re}z=0$, then trivially,

$$\|A^{\varepsilon}_{b,z}f\|_\infty\leqslant c\|f\|_1, \tag{2.2}$$

the constant c being independent of ε. Our aim then is to show that if $\operatorname{Re}z=1/2$,

$$\|A^{\varepsilon}_{b,z}f\|_2\leqslant c\left(1+|y|^{1/2}\right)\|f\|_2, \tag{2.3}$$

the constant c being independent of ε. To prove (2.3) it is enough to show that if $\operatorname{Re}z=1/2$, then

$$\left\|\hat{\tilde{K}}^{\varepsilon}_{b,z}(\xi)\right\|_\infty\leqslant c\left(1+|y|^{1/2}\right). \tag{2.4}$$

We now let $\operatorname{Re}z=1/2$ and define

$$\delta_b=\min\left((1/b)^{1/b},(3/2b)^{1/(b+1)}\right).$$

Then

$$\begin{aligned}\tilde{K}^{\varepsilon}_{b,z}(x)&=\frac{e^{i|x|^{-b}}}{|x|^{(2+b)z}}\chi\left(\varepsilon<|x|<\delta_b^{-1}\right)+\frac{e^{i|x|^{-b}}}{|x|^{(2+b)z}}\chi\left(|x|\geqslant\delta_b^{-1}\right)\\&=g(x)+h(x).\end{aligned}$$

Since $\operatorname{Re}z=1/2$ and $b>0$, $\|h\|_1\leqslant c$, and hence, $\|\hat{h}(\xi)\|_\infty\leqslant c$. We are then left with estimating $\hat{g}(\xi)$. By definition,

$$\hat{g}(\xi)=\int_{\varepsilon<|x|<\delta_b^{-1}}\frac{\exp i\left(|x|^{-b}-(2+b)y\log|x|-x\xi\right)}{|x|^{(2+b)/2}}\,dx. \tag{2.5}$$

We will henceforth write $(2+b)y$ as y.

In (2.5) we set $x=1/t$ and $1/\varepsilon=N$ to get

$$\hat{g}(\xi)=\int_{\delta_b<|t|<N}\frac{\exp i\left(|t|^b+y\log|t|-\xi t^{-1}\right)}{|t|^{1-b/2}}\,dt.$$

In the above integral we simply consider the case $t>0$, the process of estimating $t<0$ being similar. So we are left to estimate

$$\int_{\delta_b}^{N}\frac{\exp i\left(t^b+y\log t-\xi t^{-1}\right)}{|t|^{1-b/2}}\,dt. \tag{2.6}$$

We will denote $t^b + y\log t - \xi t^{-1}$ by $\psi(t)$. The estimation is done by breaking into two main cases. We pick a number $\eta > 0$ large enough so that

$$b/4 \leqslant b/3 - (b/\eta)(2b/3)^{b/(b+1)}. \tag{2.7}$$

Clearly η depends only on b.

Case 1. $|\xi| \geqslant \max(1, (\eta|y|/b)^{(b+1)/b})$. We let $t_0 = (3|\xi|/2b)^{1/(b+1)}$. If $t_0 < N$ we split (2.6) as

$$\int_{\delta_b}^{t_0} \frac{e^{i\psi(t)}}{t^{1-b/2}}\,dt + \int_{t_0}^{N} \frac{e^{i\psi(t)}}{t^{1-b/2}}\,dt = A + B. \tag{2.8}$$

If $N \leqslant t_0$, we just consider

$$\int_{\delta_b}^{N} \frac{e^{i\psi(t)}}{t^{1-b/2}}\,dt = C. \tag{2.9}$$

We note since $|\xi| \geqslant 1$, $t_0 \geqslant \delta_b$.

The estimates for A and C are similar and we do these first. We claim that in both A and C, if $\xi \geqslant 0$, then $|\psi'(t)| \geqslant c|\xi|t^{-2}$, and if $\xi \leqslant 0$, then $|\psi''(t)| \geqslant c|\xi|t^{-3}$.

Now, if $\xi \geqslant 0$, then since $t > 0$ we get

$$|\psi'(t)| = |bt^{b-1} + yt^{-1} + \xi t^{-2}| \geqslant \xi t^{-2} - |y|t^{-1} = t^{-2}(\xi - t|y|).$$

But in both A and C, $t \leqslant t_0$. Hence, we get

$$t|y| \leqslant (3/2b)^{1/(b+1)}(b/\eta)|\xi| \leqslant |\xi|/2,$$

for large η. The choice η is again compatible with (2.7). It follows then that

$$\text{if } \xi \geqslant 0, \text{ then } \quad |\psi'(t)| \geqslant c|\xi|t^{-2}, \qquad t \leqslant t_0. \tag{2.10}$$

We next consider $\xi \leqslant 0$ and compute $\psi''(t)$. We show

$$\text{if } \xi \leqslant 0, \text{ then } \quad |\psi''(t)| \geqslant c|\xi|t^{-3}, \qquad t \leqslant t_0. \tag{2.11}$$

Now, $\psi''(t) = b(b-1)t^{b-2} - yt^{-2} - 2\xi t^{-3}$. To get (2.11) we consider the two cases $b \geqslant 1$ and $0 < b < 1$.

If $b \geqslant 1$, and since $\xi \leqslant 0$, we have

$$|\psi''(t)| \geqslant 2|\xi|t^{-3} - |y|t^{-2} = t^{-3}(2|\xi| - t|y|).$$

But we have seen that if $t \leqslant t_0$, then $t|y| \leqslant |\xi|/2$. Thus $|\psi''(t)| \geqslant 3/2|\xi|t^{-3}$.

If $0 < b < 1$, clearly

$$\begin{aligned} |\psi''(t)| &\geqslant 2|\xi|t^{-3} - |y|t^{-2} - b(1-b)t^{b-2} = t^{-3}(2|\xi| - t|y| - b(1-b)t^{b+1}) \\ &\geqslant t^{-3}(2|\xi| - |\xi|/2 - b(1-b)t_0^{b+1}) \end{aligned}$$

because $t \leqslant t_0$. Substituting the value of t_0, the term on the right is bounded below by $t^{-3}(3|\xi|/2 - 3(1-b)|\xi|/2)$. Thus we see that if $t \leqslant t_0$, $|\psi''(t)| \geqslant 3b|\xi|t^{-3}/2$.

If (2.10) holds, integrating by parts A or C we get for, say, A,

$$A = -\frac{ie^{i\psi(t)}}{t^{1-b/2}\psi'(t)}\Bigg|_{\delta_b}^{t_0} - c\int_{\delta_b}^{t_0} \frac{e^{i\psi(t)}}{t^{2-b/2}}\,\frac{dt}{\psi'(t)} + c\int_{\delta_b}^{t_0} \frac{e^{i\psi(t)}}{t^{1-b/2}}\,\frac{\psi''(t)\,dt}{(\psi'(t))^2}.$$

For C, we replace t_0 by N at each occurrence above. In both cases, though, $t \leqslant t_0$. We now use the fact that $|\psi'(t)| \geqslant c|\xi|t^{-2}$ to get

$$|A| \leqslant c|\xi|^{-1}\left(1 + t_0^{1+b/2}\right) + c|\xi|^{-1}\int_{\delta_b}^{t_0} t^{b/2}\,dt$$
$$+ c|\xi|^{-2}\int_{\delta_b}^{t_0}\left(t^{1+3b/2} + |y|t^{1+b/2} + |\xi|t^{b/2}\right)dt.$$

The term C is also majorized by the term on the right side above. Using the fact that $t|y| \leqslant c|\xi|$, if $t \leqslant t_0$, and substituting the value for t_0 after performing the indicated intègrations, we get

$$|A| \leqslant c|\xi|^{-1} + c|\xi|^{-b/2(b+1)}.$$

Since $|\xi| \geqslant 1$, it follows that $|A|$ and $|C|$ are both bounded by a constant.

We now consider the situation (2.11). We define $\zeta(v) = \int_{\delta_b}^{v} e^{i\psi(t)}\,dt$. By Van der Corput's lemma [**18**, Volume 1, p. 197] and because (2.11) holds, $|\zeta(v)| \leqslant c|v|^{3/2}|\xi|^{-1/2}$. Integrating A or C by parts we get, say, for A,

$$A = \left.\frac{\zeta(t)}{t^{1-b/2}}\right|_{\delta_b}^{t_0} - c\int_{\delta_b}^{t_0}\frac{\zeta(t)}{t^{2-b/2}}\,dt.$$

For C, we replace t_0 by N at each occurrence above. We now substitute the estimate for $\zeta(t)$ obtained above, to get

$$|A| \leqslant c|\xi|^{-1/2} + \frac{t_0^{(b+1)/2}}{|\xi|^{1/2}} + c|\xi|^{-1/2}\int_{\delta_b}^{t_0} t^{b/2-1/2}\,dt.$$

Substituting the value for t_0 and using $|\xi| \geqslant 1$, we conclude that $|A|$ and $|C|$ are both bounded by a constant.

We now estimate B. We claim here that since $t_0 < t < N$, $|\psi'(t)| \geqslant ct^{b-1}$. To see this we note

$$|\psi'(t)| = |bt^{b-1} + yt^{-1} + \xi t^{-2}| \geqslant t^{b-1}\left(b - |y|t^{-b} - |\xi|t^{-(b+1)}\right),$$

but since $t_0 < t$ and $b > 0$,

$$b - |y|t^{-b} - |\xi|t^{-(b+1)} \geqslant b - |y|t_0^{-b} - |\xi|t_0^{-(b+1)}.$$

Substituting in the value of t_0 and using $|y| \leqslant (b/\eta)|\xi|^{b/(b+1)}$, we get

$$b - |y|t_0^{-b} - |\xi|t_0^{-(b+1)} \geqslant b - (b/\eta)(2b/3)^{b/(b+1)} - 2b/3.$$

In view of the choice of η, as determined by (2.7), it follows that $|\psi'(t)| \geqslant ct^{b-1}$.

We now estimate B. Integrating by parts we get

$$B = -\left.\frac{ie^{i\psi(t)}}{t^{1-b/2}\psi'(t)}\right|_{t_0}^{N} - c\int_{t_0}^{N} e^{i\psi(t)}\left(\frac{1}{t^{2-b/2}\psi'(t)} + \frac{\psi''(t)}{t^{1-b/2}(\psi'(t))^2}\right)dt.$$

Substituting the estimate for $\psi'(t)$ obtained above, we get

$$|B| \leqslant \frac{c}{t_0^{b/2}} + c\int_{t_0}^{\infty}\frac{dt}{t^{1+b/2}} + c\int_{t_0}^{\infty}\frac{1}{t^{3b/2}}\left(\frac{|y|}{t} + \frac{|\xi|}{t^2} + \frac{1}{t^{1-b}}\right)dt.$$

Using the fact that $|y| \leqslant (b/\eta)|\xi|^{b/(b+1)}$, $|\xi| \geqslant 1$, and the value for t_0, we easily see that $|B| \leqslant ct_0^{-b/2} \leqslant c$.

We now consider the next case that arises.

Case 2. $|\xi| \leqslant \max(1, (\eta|y|/b)^{(b+1)/b})$. We pick a number $\alpha > 1$; α will depend only on b and its choice will be made later. We then define

$$t_0 = \max\left((\alpha/b)^{1/b}, (\alpha\eta|y|/b^2)^{1/b}\right).$$

Note that $t_0 > \delta_b$. We split (2.6) exactly as in (2.8) and (2.9), dependent on $t_0 < N$ or $N \leqslant t_0$. We dispose of A and C by simply noting that both A and C are majorized by

$$\int_{\delta_b}^{t_0} \frac{dt}{t^{1-b/2}} \leqslant c\left(t_0^{b/2} + \delta_b^{b/2}\right) \leqslant c\left(1 + |y|^{1/2}\right).$$

We now estimate B. We shall show that if $t_0 < t < N$, then $|\psi'(t)| \geqslant ct^{b-1}$. This will be shown by considering the two cases $\eta|y| \leqslant b$ and $\eta|y| > b$. Recall now that $\psi'(t) = bt^{b-1} + yt^{-1} + \xi t^{-2}$.

We first consider the case $\eta|y| \leqslant b$. Then by the definition of t_0, $t_0 = (\alpha/b)^{1/b}$. Thus we get

$$|\psi'(t)| \geqslant t^{b-1}\left(b - |y|t^{-b} - |\xi|t^{-(b+1)}\right).$$

But since $t_0 < t < N$,

$$|\psi'(t)| \geqslant t^{b-1}\left(b - |y|t_0^{-b} - |\xi|t_0^{-(b+1)}\right). \tag{2.12}$$

Moreover, if $\eta|y| \leqslant b$, it follows that $|\xi| \leqslant 1$, hence

$$|\psi'(t)| \geqslant t^{b-1}\left(b - b^2/(\alpha\eta) - (b/\alpha)^{(b+1)/b}\right).$$

A suitable large value of α then ensures that $b^2/(\alpha\eta) + (b/\alpha)^{(b+1)/b} \leqslant b/2$. It then follows that $|\psi'(t)| \geqslant ct^{b-1}$.

We now let $\eta|y| > b$. Then $t_0 = (\alpha\eta|y|/b^2)^{1/b}$ and $|\xi| \leqslant (\eta|y|/b)^{(b+1)/b}$. Using these estimates in (2.12) we get

$$|\psi'(t)| \geqslant t^{b-1}\left(b - |y|t_0^{-b} - (\eta|y|/b)^{(b+1)/b}t_0^{-(b+1)}\right).$$

The term in brackets on the right side is bounded below by $b - b^2/(\alpha\eta) - (b/\alpha)^{(b+1)/b} \geqslant b/2$. So we have now proved that $|\psi'(t)| \geqslant ct^{b-1}$.

Hence, integrating B by parts we get

$$B = -\left.\frac{ie^{i\psi(t)}}{t^{1-b/2}\psi'(t)}\right|_{t_0}^{N} + c\int_{t_0}^{N} e^{i\psi(t)}\left(\frac{1}{t^{2-b/2}\psi'(t)} + \frac{\psi''(t)}{t^{1-b/2}(\psi'(t))^2}\right)dt.$$

Hence, using the estimate $|\psi'(t)| \geqslant ct^{b-1}$ we get

$$|B| \leqslant ct_0^{-b/2} + c\int_{t_0}^{\infty}\left(\frac{1}{t^{1+b/2}} + \frac{|y|}{t^{1+3b/2}} + \frac{|\xi|}{t^{2+3b/2}}\right)dt.$$

Using the value for t_0 and performing the integrations, it follows that the right side is bounded by a constant. This proves (2.4) and hence (2.3). Using the result of **[13]** it follows that for fixed $\varepsilon > 0$, and $z = 1/p$, we have by interpolating between (2.2) and (2.3),

$$\left\|\tilde{K}^{\varepsilon}_{b,p} * f\right\|_p \leqslant c_p \|f\|_{p'}, \qquad 1/p + 1/p' = 1. \tag{2.13}$$

Up to this stage we have not made use of the hypothesis $(b + 2)/p < 1$. We shall now make use of it for the limiting argument which will yield the lemma. Pick $f \in L^{p'}$, $p' \neq 1$. Note that it is only necessary to consider the case $p' \neq 1$. Now $f = f_1 + f_2$, where $\|f_2\|_{p'} < \delta$, $\delta > 0$ any preassigned number and $f_1 \in C_0^\infty(R)$. We now show that $\tilde{K}^{\varepsilon}_{b,p} * f$ is a Cauchy sequence in the norm of L^p. For $\varepsilon_1 < \varepsilon_2$, then

$$\begin{aligned} &\left\|\tilde{K}^{\varepsilon_1}_{b,p} * f - \tilde{K}^{\varepsilon_2}_{b,p} * f\right\|_p \\ &\quad \leqslant \left\|\left(\tilde{K}^{\varepsilon_1}_{b,p} - \tilde{K}^{\varepsilon_2}_{b,p}\right) * f_1\right\|_p + \left\|\tilde{K}^{\varepsilon_1}_{b,p} * f_2\right\|_p + \left\|\tilde{K}^{\varepsilon_2}_{b,p} * f_2\right\|_{p}. \end{aligned} \tag{2.14}$$

But

$$\tilde{K}^{\varepsilon_1}_{b,p}(x) - \tilde{K}^{\varepsilon_2}_{b,p}(x) = \frac{e^{i|x|^{-b}}}{|x|^{(b+2)/p}} \chi(\varepsilon_1 < |x| < \varepsilon_2).$$

Hence, $\|\tilde{K}^{\varepsilon_1}_{b,p}(x) - \tilde{K}^{\varepsilon_2}_{b,p}(x)\|_1 \leqslant c\varepsilon_2^{1-(b+2)/p}$ since $(b + 2)/p < 1$. Hence, the right side of (2.14) may be estimated by $2\delta + \varepsilon_2^{1-(b+2)/p}\|f_1\|_p$. Hence, $\tilde{K}^{\varepsilon}_{b,p} * f(x)$ is convergent in L^p norm. This proves our lemma. Q.E.D.

REMARKS. 1. For purposes of application it is enough to prove Lemma (2.1) for the truncated kernels

$$\tilde{K}^{\varepsilon}_{b,p}(x) = \frac{e^{i|x|^{-b}}}{|x|^{(b+2)/p}} \chi(\varepsilon < |x| < 1).$$

2. Lemma (2.1) is valid in n-dimensions too. The kernels we consider then are

$$\tilde{K}_{b,p}(x) = \frac{e^{i|x|^{-b}}}{|x|^{n(b+2)/p}}.$$

Since these are radial functions, we can use the formulas for the Fourier transform in Chapter 4 of **[13]**.

The computation, however, follows the same outline as above. The point to observe is that one makes the choice of t_0 at those points where the phase $\psi(t)$ is stationary.

3. By a simple change of variable and from Lemma (2.1) it follows that, if we let

$$\tilde{K}_{b,p}(x) = \frac{e^{i\alpha_b|x|^{-b}}}{|x|^{(b+2)/p}}, \qquad \alpha_b \neq 0 \text{ and real},$$

then for $(b + 2)/p < 1$, $0 < b < \infty$,

$$\left\|\tilde{K}_{b,p} * f\right\|_p \leqslant c_p \|f\|_{p'}, \qquad 1/p + 1/p' = 1.$$

We will now use the lemma above to prove the basic estimate which will yield Theorem A. For $\alpha_b \neq 0$ and real, $x \in R$, recall that

$$K_b(x) = \frac{e^{i\alpha_b|x|^{-b}}}{|x|} \chi(|x| \leqslant 1), \qquad 0 < b < \infty.$$

LEMMA (2.15). *Let* $f(x) \in C_0^\infty(R)$. *Then, for* $r > 1$,

$$(K_b * f)^{\#}(x) \leqslant c_{r,b}(|f|^r)^{*1/r}(x).$$

PROOF. Since the operators we are dealing with are convolution operators it will be enough to verify the above estimate for an interval centered at the origin. Let this interval be called I, and let $|I| = \delta$. We fix a number, $\delta_0(b) > 0$, such that $\delta_0 + \delta_0^{1/(b+1)} \leqslant 1/8$ and $2\delta_0 \leqslant \delta_0^{1/(b+1)}$. This is to avoid some trivial technical problems. Moreover, it is clearly enough to prove the lemma for r, near to 1, i.e. we assume that $0 < (b+2)/r' < 1$, where $1/r + 1/r' = 1$.

Case 1. $\delta \geqslant \delta_0$. We let $f(x) = f_1(x) + f_2(x)$, $f_1(x) = f(x)\chi_{(20/\delta_0)I}(x)$. It follows that

$$\int_I |K_b * f|\,dx \leqslant \int_I |K_b * f_1|\,dx + \int_I |K_b * f_2(x)|\,dx.$$

By the location of the support of $f_2(x)$, the last term vanishes. For the first term, on the right we have

$$\int_I |K_b * f_1|\,dx \leqslant \left(\int_R |K_b * f_1|^r dx\right)^{1/r} \delta^{1/r'}; \qquad 1/r + 1/r' = 1.$$

By (0.1) the right side above is majorized by

$$\left(\int_{(20/\delta_0)I} |f_1|^r dx\right)^{1/r} \delta^{1/r'} \leqslant c\delta(|f|^r)^{*1/r}(0).$$

Hence, $\delta^{-1}\int_I |K_b * f|\,dx \leqslant c(|f|^r)^{*1/r}(0)$. This verifies the lemma in this case.

Case 2. $\delta \leqslant \delta_0$. Let $f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x)$,

$$f_1(x) = f(x)\chi_{2I}(x), f_2(x) = f(x)\chi\{x: 2\delta \leqslant |x| \leqslant \delta^{1/(b+1)}\}(x),$$

$$f_3(x) = f(x)\chi\{x: \delta^{1/(b+1)} < |x| \leqslant 1/2\}(x), \quad f_4(x) = f(x)\chi\{1/2 < |x| < 4\}(x).$$

We estimate $f_1(x)$ first:

$$\int_I |K_b * f_1|\,dx \leqslant \left(\int_R |K_b * f_1|^r dx\right)^{1/r} \delta^{1/r'}$$
$$\leqslant c\left(\int_{2I} |f_1|^r dx\right)^{1/r} \delta^{1/r'} \leqslant c\delta(|f|^r)^{*1/r}(0).$$

Hence we get the desired estimate. We now note that if $x \in I$, due to the location of the support of $f_5(x)$, $K_b * f_5(x) = 0$. Now using the fact that $|K_b(x)| \leqslant 1/|x|$, it follows that for any $x \in I$, $|K_b * f_4(x)| \leqslant cf^*(0)$. Hence we only have to estimate $f_3(x)$ and $f_2(x)$. Define

$$c_I = \int \frac{e^{i\alpha_b|t|^{-b}}}{|t|} f_3(t)\,dt.$$

Note that because $f(x)$ is locally integrable and by the location of the support of $f_3(t)$, c_I is a finite number. For $x \in I$,

$$(2.16)\quad |K_b * f_3(x) - c_I|$$
$$\leqslant \int |f_3(t)| \left| \frac{e^{i\alpha_b|x-t|^{-b}}}{|x-t|}\chi(|x-t| \leqslant 1) - \frac{e^{i\alpha_b|t|^{-b}}}{|t|}\chi(|t| \leqslant 1)\right| dt.$$

But if t lies in the support of f_3, and $x \in I$,

$$\left| \frac{e^{i\alpha_b|x-t|^{-b}}}{|x-t|}\chi(|x-t| \leqslant 1) - \frac{e^{i\alpha_b|t|^{-b}}}{|t|}\chi(|t| \leqslant 1)\right|$$
$$= \left| \frac{e^{i\alpha_b|x-t|^{-b}}}{|x-t|} - \frac{e^{i\alpha_b|t|^{-b}}}{|t|}\right| \leqslant c\left(|x|/|t|^{b+2}\right).$$

The last estimate follows from the mean value theorem and because $2|x| \leqslant |t| \leqslant 1$. Substituting the last estimate in (2.16) we get

$$|K_b * f_3(x) - c_I| \leqslant c\int |f_3(x)| \frac{|x|}{|t|^{b+2}}\, dt \leqslant c\delta \int \frac{|f_3(t)|}{|t|^{b+2}}\, dt$$
$$\leqslant c\delta \sum_{k=0}^{\infty} \frac{1}{(2^k\delta)^{(b+2)/(b+1)}} \int_{2^k\delta^{1/(b+1)} \leqslant |t| < 2^{k+1}\delta^{1/(b+1)}} |f_3(t)|\, dt$$
$$\leqslant cf^*(0).$$

Hence, $(I/|I|)\int_I |K_b * f_3(x) - c_I|\,dx \leqslant cf^*(0)$.

We now estimate $f_2(x)$. Now

$$K_b * f_2(x) = \int \frac{e^{i\alpha_b|x-t|^{-b}}}{|x-t|} f_2(t)\, dt.$$

We split the above into two terms as follows:

$$K_b * f_2(x) = \int \frac{e^{i\alpha_b|x-t|^{-b}}}{|x-t|^{(2+b)/r'}} \left(\frac{1}{|x-t|^{1-(2+b)/r'}} - \frac{1}{|t|^{1-(2+b)/r'}} \right) f_2(t)\, dt$$
$$+ \int \frac{e^{i\alpha_b|x-t|^{-b}}}{|x-t|^{(2+b)/r'}} \left(\frac{f_2(t)}{|t|^{1-(2+b)/r'}} \right) dt$$
$$= A + B.$$

Using the mean value theorem for the term in brackets in the integrand of A, we get, by noting $|t| \geqslant 2|x|$, that

$$|A| \leqslant c|x| \int_{|t|>\delta} \frac{|f_2(t)|}{|t|^2}\, dt \leqslant c\delta \sum_{k=0}^{\infty} \frac{1}{(2^k\delta)^2} \int_{|t| \sim 2^k\delta} |f(t)|\, dt \leqslant cf^*(0).$$

Now $B = \tilde{K}_{b,r'} * (f_2(t)/|t|^{1-(2+b)/r'})(x)$. Here

$$\tilde{K}_{b,r'}(x) = \frac{e^{i\alpha_b|x|^{-b}}}{|x|^{(2+b)/r'}}.$$

Hence, by inserting the estimates for A and B, we have

$$(2.17)\qquad \int_I |K_b * f_2|\,dx \leqslant c\delta f^*(0) + \int_I \left|\tilde{K}_{b,r'} * \left(\frac{f_2(t)}{|t|^{1-(2+b)/r'}}\right)(x)\right| dx.$$

We estimate the second term on the right above as follows:

$$\int_I \left|\tilde{K}_{b,r'} * \left(\frac{f_2(t)}{|t|^{1-(2+b)/r'}}\right)(x)\right| dx \leqslant \delta^{1/r}\left(\int_R \left|\tilde{K}_{b,r'} * \left(\frac{f_2(t)}{|t|^{1-(2+b)/r'}}\right)(x)\right|^{r'} dx\right)^{1/r'}.$$

We now use Lemma (2.1) (see Remark 3, following Lemma (2.1) too) to estimate the right side above to get

$$(2.18)\qquad \leqslant c\delta^{1/r}\left(\int \frac{|f_2(t)|^r}{|t|^{r-(2+b)(r-1)}}\,dt\right)^{1/r}$$

$$\leqslant c\delta^{1/r}\left[\sum_{k=0}^{k_0} (2^k\delta)^{(r-1)(b+1)} \frac{1}{2^k\delta}\int_{|t|\sim 2^k\delta} |f(t)|^r dt\right]^{1/r}.$$

Here k_0 is an integer such that $2^{k_0-1}\delta < \delta^{1/(b+1)} \leqslant 2^{k_0}\delta$. We thus estimate (2.18) by

$$c\delta^{1/r}\left(\sum_{k=0}^{k_0} (2^k\delta)^{(r-1)(b+1)}\right)^{1/r} (|f|^r)^{*1/r}(0)$$

$$\leqslant c\delta^{1/r}\delta^{(r-1)(b+1)/r(b+1)}(|f|^r)^{*1/r}(0) \leqslant c\delta(|f|^r)^{*1/r}(0).$$

Hence (2.17) is majorized by $c\delta(|f|^r)^{*1/r}(0)$, the desired estimate. Q.E.D.

REMARK. It is evident that the proof given above may be easily adapted to R^n.

LEMMA (2.19). *Let* $w \in A_p$. *Then, if* $1 < p < \infty$, $\|K_b * f\|_{p,w} \leqslant c_p\|f\|_{p,w}$.

PROOF. Our aim is to use Theorems (1.4) and (1.5). We recall that if $w \in A_p$ then $w \in A_\infty$. With no loss of generality we will assume that $f \in C_0^\infty$. The lemma then follows by a standard density argument. Note that to apply Theorem (1.5) we need to show that $\|K_b * f\|_{p,w} < \infty$. Now we recall that if $w \in A_p$, then by Theorem (1.6) there exists a number r, $1 < r < \infty$, such that $\int_I w^r(x)dx < \infty$, where I is any interval. Hence, noting that $K_b(x)$ has compact support and $f(x)$ has compact support, $K_b * f(x)$ has its support in some interval I. Thus, by Theorem (1.4),

$$\|K_b * f\|_{p,w} \leqslant c_p\int_I |K_b * f|^p w(x)\,dx \leqslant c_p\left(\int_R |K_b * f|^{pr'}dx\right)^{1/r'}\left(\int_I w^r(x)\,dx\right)^{1/r}$$

$$\leqslant c\left(\int |f|^{pr'}dx\right)^{1/r'} < \infty,$$

where $1/r + 1/r' = 1$. Hence, applying Theorems (1.4) and (1.5) we have for $w \in A_\infty$,

$$\|K_b * f\|_{p,w} \leqslant c_p\|(K_b * f)^\#\|_{p,w}, \qquad 1 < p < \infty.$$

But applying Lemma (2.15) to the right side above we get

$$\|K_b * f\|_{p,w} \leqslant c_p\|(|f|^r)^{*1/r}\|_{p,w}.$$

Choosing r sufficiently close to one, we immediately get, by Theorem (1.4),

$$\|K_b * f\|_{p,w} \leqslant c_p\|f\|_{p,w}, \qquad 1 < p < \infty.$$

This is the desired result. Q.E.D.

We now are in a position to give a proof of Theorem A of the Introduction.

PROOF OF THEOREM A. Using Lemma (1.2) we see

$$\|T_b f\|_{p,w} \leqslant c_p\|K_{b'} * f\|_{p,w} + \|f^*\|_{p,w}.$$

Applying Lemma (2.19) to the first term on the right side, and Theorem (1.4) to the second term it follows immediately that $\|T_b f\|_{p,w} \leqslant c\|f\|_{p,w}$, $1 < p < \infty$. Q.E.D.

3. Preliminary lemmas for the weak (1,1) estimate. The aim of this section will be to bring together the ingredients needed to prove Theorem C of the Introduction. The emphasis of the proof, as pointed out before, will be on the multiplier. The first result we wish to recall is a theorem of R. Coifman and R. Rochberg [**3**].

THEOREM (3.1). *Let $w(x) \in A_1$. Then there exists σ, $0 < \sigma < 1$, and functions $g(x)$ and $h(x)$ such that $w(x) = h(x)(g^*(x))^\sigma$. The function $h(x)$ is essentially a constant, i.e. there is a number $c > 0$, such that $c^{-1} < h(x) < c$. Moreover, if $g(x)$ is any locally integrable function such that $g^*(x) < \infty$ a.e. and $c^{-1} < h(x) < c$, then for any σ, $0 < \sigma < 1$, $h(x)(g^*(x))^\sigma$ is in A_1.*

We now wish to write down some results for weighted H^1 spaces which will prove crucial in the subsequent development. The sequence of Theorems (3.2)–(3.4) is due to Strömberg and Torchinsky and may be found in [**14–16**].

THEOREM (3.2) (ANALYTIC INTERPOLATION). *Let $w(x) \in A_1$. Let T_z be an analytic family of operators in the following sense. For $g \in L^\infty$ with compact support, $f(x) \in L^2 \cap H^1_w$, consider*

$$U(z) = \int_{R^n} T_z f(x) \frac{g(x)}{|g(x)|} |g(x)|^{p'(1-z)} (w_\varepsilon(x))^{1-p(1-z)} dx.$$

Here $1/p + 1/p' = 1$ and $w_\varepsilon(x) = w(x)\chi_E(x)$, $E = \{x\colon \varepsilon < w(x) < 1/\varepsilon\}$. Suppose $U(z)$ is an analytic function for $z \in S$, the strip in the complex plane $= \{z\colon 1/2 < \operatorname{Re} z < 1\}$, and let $U(z)$ be continuous for $\operatorname{Re} z = 1$ and $\operatorname{Re} z = 1/2$. Moreover, assume that, for some $N \geqslant 0$,

$$\|T_z f\|_{1,w} \leqslant c(1 + |y|^N)\|f\|_{H^1_w}, \qquad \operatorname{Re} z = 1,$$

and

$$\|T_z f\|_2 \leqslant c\|f\|_2, \qquad \operatorname{Re} z = 1/2;$$

then if $\operatorname{Re} z = 1/p$, $1 < p \leqslant 2$, we have

$$\|T_z f\|_{p,w^{2-p}} \leqslant c_p\|f\|_{p,w^{2-p}}.$$

We remark that the result of Strömberg and Torchinsky is much more general than the statement here. However, we need only this special case of the general theorem of [**16**]. For example, the above result is valid for $w \in A_\infty$, provided we replace the norm on the right side in the conclusion by the $H^p_{w^{2-p}}$ norm.

The next result we need is the atomic decomposition of H_w^1. An atom is a function $a(x)$ such that:

(1) $a(x)$ is supported in a cube I,

(2) $\int_I a(x)\,dx = 0$, and

(3) $\|a\|_\infty \leqslant 1/w(I)$.

Then from **[15]** we have the following result.

THEOREM (3.3). *Let $w(x) \in A_1$. Then if $f \in H_w^1$, there is a sequence of atoms $\{a_k\}$ such that $f \sim \Sigma\lambda_k a_k$ and $\|f\|_{H_w^1} \sim \Sigma|\lambda_k|$.*

We need one more fact on the imaginary fractional integral. For $f \in \hat{C}_{0,0}^\infty(R^n)$ and $y \in R$, define $\widehat{R_y f}(\xi) = |\xi|^{iy}\hat{f}(\xi)$. Since $|\xi|^{iy} = e^{iy\log|\xi|}$, the multiplier operator R_y is a Hörmander multiplier. Then from **[14]** we have the next result.

THEOREM (3.4). *Let $w(x) \in A_1$. Then*

$$\|R_y f\|_{H_w^1} \leqslant c(1 + |y|^N)\|f\|_{H_w^1}.$$

The constant N depends only on the dimension and is positive.

Theorem (3.4) stated here is a special case of a general multiplier result in **[14]**.

The next aim will be to employ the results above to prove the following lemma.

LEMMA (3.5). *Let $w \in A_1$. Then there exists $p_0 = p_0(w)$, $1 < p_0 \leqslant 2$, such that $\|T_{b,a}f\|_{p_0,w} \leqslant c\|f\|_{p_0,w}$ with $a = b(1/p_0 - 1/2)$.*

Before we go about proving this lemma, we wish to remark that in n-dimensions the above result is also valid except that the value of $a = nb(1/p_0 - 1/2)$. The proof of the lemma is atomic. The first step hence is the next lemma.

LEMMA (3.6). *Let $v \in A_1$. Then $\|T_b f\|_{1,v} \leqslant c\|f\|_{H_v^1}$.*

PROOF. Let $a(x)$ be an atom in H_v^1, supported in an interval I, centered at x_0. To prove the lemma it is enough to show that $\|T_b a\|_{1,v} \leqslant c$. Since $v \in A_1$, by Theorem (1.6) we get

$$\left(\frac{1}{|I|}\int_I v^{p'}\,dx\right)^{1/p'} \leqslant \frac{c}{|I|}\int_I v\,dx \tag{3.7}$$

for some p', $1 < p' < \infty$, and all intervals I. Choose p such that $1/p + 1/p' = 1$. The estimates break up into two cases. We choose a number $\delta_0 > 0$ dependent only on b, by $2\delta_0 < \delta_0^{1-b}$, $2\delta_0 + \delta_0^{1-b} \leqslant 1/8$.

Case 1. $|I| \geqslant \delta_0$. This is the trivial case. Let $I^* = (10/\delta_0)I$. Now

$$\int_R |T_b a| v\,dx \leqslant \int_{I^*} |T_b a| v\,dx + \int_{R\setminus I^*} |T_b a| v\,dx = A + B.$$

The estimates for B are trivial ones. From (1.1) we see that, if $x \in R\setminus I^*$,

$$|T_b a(x)| \leqslant \frac{c}{|x - x_0|^2}\int_I |a(t)|\,dt \leqslant c\left(\frac{|I|}{v(I)}\right)\left(\frac{1}{|x - x_0|^2}\right).$$

Substituting this estimate in B we get

$$B \leqslant \left(\frac{c|I|}{v(I)}\right)\int_{|x-x_0|>|I|}\frac{v(x)}{|x-x_0|^2}\,dx \leqslant \frac{c|I|}{v(I)}\sum_{k=0}^{\infty}\frac{1}{(2^k|I|)^2}\int_{|x-x_0|\sim 2^k|I|} v\,dx.$$

Since $v \in A_1$, the last term is majorized by

$$\frac{c|I|}{v(I)}\sum_{k=0}^{\infty}\left(\frac{v(I)}{|I|}\right)\left(\frac{1}{2^k|I|}\right).$$

Using the fact that $|I| \geqslant 2$, the above term is bounded above by a constant. To estimate A,

$$A \leqslant \left(\int_{I^*}|T_b a|^p\,dx\right)^{1/p}\left(\frac{1}{|I^*|}\int_{I^*} v^{p'}\,dx\right)^{1/p'}.$$

Using (0.1), (3.7) and the fact that v is doubling, the right side may be estimated by

$$c\left(\int_R |a|^p\,dx\right)^{1/p}\left(\frac{1}{|I|}\int_I v\,dx\right) \leqslant c.$$

Case 2. $|I| \leqslant \delta_0$. To estimate this case we use the kernel estimates. Let $|I| = \delta$, and $J =$ the interval concentric with I, but with length δ^{1-b}. Let $I^* = 2I$. Now

$$\int_R |T_b a| v\,dx \leqslant \int_{I^*}|T_b a| v\,dx + \int_{R\setminus I^*}|T_b a| v\,dx = A + B.$$

Now

$$A \leqslant \left(\int_R |T_b a|^p\,dx\right)^{1/p}\left(\int_{I^*} v^{p'}\,dx\right)^{1/p'} \leqslant c\left(\int_R |a|^p\,dx\right)^{1/p}\frac{v(I)}{|I|^{1/p}} \leqslant c.$$

We now estimate the term B. Now note by (1.1), for $x \in R\setminus I^*$, and $b' = b/(1-b)$ we get

$$\begin{aligned}|T_b a(x)| &\leqslant c|K_{b'} * a(x)| + c\int_I |a(t)|\left(\frac{1}{(|1+|x-x_0|)^2} + \frac{\chi(|x-x_0|\leqslant 2)}{|x-x_0|^{1-\varepsilon}}\right)dt \\ &\leqslant c|K_{b'} * a(x)| + \frac{c|I|}{v(I)}\left((1+|x-x_0|)^{-2} + \chi(|x-x_0|\leqslant 2)|x-x_0|^{-1+\varepsilon}\right).\end{aligned}$$

But since $|I| \leqslant 1$ and $v \in A_1$, the second term, when substituted into the integrand of B, yields

$$\frac{c|I|}{v(I)}\int_{R\setminus I^*}\left((1+|x-x_0|)^{-2} + \chi(|x-x_0|\leqslant 2)|x-x_0|^{-1+\varepsilon}\right)v(x)\,dx \leqslant c.$$

Consequently, we are only left to estimate $|K_{b'} * a(x)|$ for $x \in R\setminus I^*$. Now

$$\int_{R\setminus I^*}|K_{b'} * a| v\,dx \leqslant \int_{J\setminus I^*}|K_{b'} * a| v\,dx + \int_{R\setminus J}|K_{b'} * a| v\,dx = C + D.$$

The technique we use to estimate the terms C and D is the same as for the sharp function estimates of Lemma (2.15). By the moment conditions on $a(x)$, if $x \in R\setminus J$,

$$K_{b'} * a(x) = c\int\left(\frac{e^{i\alpha_b|x-y|^{-b'}}}{|x-y|} - \frac{e^{i\alpha_b|x-x_0|^{-b'}}}{|x-x_0|}\right)a(y)\,dy.$$

Clearly then, by the mean value theorem,

$$|K_{b'} * a(x)| \leqslant \frac{c\delta}{|x-x_0|^{b'+2}} \int |a(y)|dy \leqslant \frac{c|I|^2}{|x-x_0|^{(b'+2)}}\left(\frac{1}{v(I)}\right).$$

Hence,

$$D \leqslant \frac{c|I|^2}{v(I)} \int_{|x-x_0|>\delta^{1-b}} \frac{v(x)}{|x-x_0|^{(b'+2)}}\,dx$$

$$\leqslant \frac{c|I|^2}{v(I)} \sum_{k=k_0}^{\infty} \frac{1}{(2^k\delta)^{(b'+1)}}\left(\frac{1}{(2^k\delta)} \int_{|x-x_0|\leqslant 2^k\delta} v\,dx\right),$$

k_0 is a positive integer such that $2^{k_0} \leqslant \delta^{1-b} < 2^{k_0+1}\delta$. Using the fact that $b' = b/(1-b)$ and $v \in A_1$, the above is easily seen to be majorized by a constant. We now estimate the term C. By the choice of δ_0, the term C does arise. For $x \in J \setminus I^*$, we split $K_{b'} * a(x)$ as follows:

$$K_{b'} * a(x) = c\int \frac{e^{i\alpha_b|x-y|^{-b'}}}{|x-y|^{(b'+2)/p}}\left[\frac{1}{|x-y|^{1-(b'+2)/p}} - \frac{1}{|x-x_0|^{1-(b'+2)/p}}\right]a(y)\,dy$$

$$+c\left(\tilde{K}_{b',p} * a(x)\right)\frac{1}{|x-x_0|^{1-(b'+2)/p}} = E + F.$$

Applying the mean value theorem to the term in brackets in the integrand of E, and noting that for $y \in I$, and $x \in J \setminus I^*$, $|x-y| \geqslant c|x-x_0|$, we have

$$|E| \leqslant \frac{c|I|}{|x-x_0|^2} \int_I |a(y)|dy \leqslant \frac{c|I|^2}{|x-x_0|^2}\left(\frac{1}{v(I)}\right).$$

Substituting the estimate above for E and the term F in C we have

$$|C| \leqslant \frac{c|I|^2}{v(I)} \int_{|x-x_0|>\delta} \frac{v(x)}{|x-x_0|^2}\,dx + c\int_{\delta \leqslant |x-x_0| \leqslant \delta^{1-b}} |\tilde{K}_{b',p} * a| \frac{v(x)}{|x-x_0|^{1-(b'+2)/p}}\,dx.$$

By a standard argument since $v \in A_1$, the first term is majorized by a constant. For the second term we use Hölder's inequality to get

$$\int_{|x-x_0|>\delta} |\tilde{K}_{b',p} * a| \frac{v(x)}{|x-x_0|^{1-(b'+2)/p}}\,dx$$

$$\leqslant \left(\int_R |\tilde{K}_{b',p} * a|^p\,dx\right)^{1/p}\left(\int_{\delta<|x-x_0|<\delta^{1-b}} \frac{v^{p'}(x)}{|x-x_0|^{p'-(b'+2)(p'-1)}}\,dx\right)^{1/p'}.$$

Note that p' may be chosen close to one; this means that $(b'+2)/p < 1$ and Lemma (2.1) applies on the first term above. Hence, the term on the right is majorized by

$$c\left(\int |a|^{p'}\,dx\right)^{1/p'}\left(\sum_{k=0}^{k_0} (2^k\delta)^{(p'-1)(b'+1)} \frac{1}{2^k\delta} \int_{|x-x_0|\leqslant 2^k\delta} v^{p'}\,dx\right)^{1/p'}.$$

Here, $2^{k_0-1} < \delta^{1-b} \leqslant 2^{k_0}\delta$. Now using $b' = b/(1-b)$, and (3.7), the above is majorized by

$$c\frac{|I|^{1/p'}}{v(I)}\left(\frac{v(I)}{|I|}\right)\left(\sum_{k=0}^{k_0}(2^k\delta)^{1/(1-b)}\right)^{1/p'} \leqslant c.$$

This proves Lemma (3.6). Q.E.D.

PROOF OF LEMMA (3.5). Since $w \in A_1$, by Theorem (3.1), $w(x) = h(x)(g^*(x))^\sigma$, $0 < \sigma < 1$. With no loss of generality, to prove Lemma (3.5) we may take $h(x) \equiv 1$. Pick a number η such that $\sigma < \eta < 1$. Consider the weight $(g^*(x))^\eta = v(x)$. The weight $v \in A_1$ by Theorem (3.1). Now there exists a p_0, $1 < p_0 \leqslant 2$, such that $(2-p_0)\eta = \sigma$. This is the choice of p_0 for Lemma (3.5). Let us now define the family of operators $\{T_{b,z}\}$ for $f \in \hat{C}_{0,0}^\infty$ by

$$\widehat{T_{b,a}f}(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^{bz-b/2}}\hat{f}(\xi).$$

We now wish to see that the family $\{T_{b,z}\}$ is analytic in the sense of Theorem (3.2). We have

$$U(z) = \int_R (T_{b,z}f(x))\frac{g(x)}{|g(x)|}|g(x)|^{p_0'(1-z)}(v_\varepsilon(x))^{1-p_0(1-z)}\,dx.$$

If $f \in L^2 \cap H_v^1$ and $g(x)$ has compact support, note the integral above converges absolutely. If $1/2 < \operatorname{Re} z < 1$, by differentiating in z under the integral we get

$$U'(z) = \int_R \frac{d}{dz}(T_{b,z}f)(x)\frac{g(x)}{|g(x)|}|g(x)|^{p_0'(1-z)}(v_\varepsilon(x))^{1-p_0(1-z)}\,dx$$
$$+\int_R T_{b,z}f(x)\frac{g(x)}{|g(x)|}\frac{d}{dz}\left(|g(x)|^{p_0'(1-z)}(v_\varepsilon(x))^{1-p_0(1-z)}\right)dx.$$

If $f \in L^2$ and $1/2 < \operatorname{Re} z < 1$, $(d/dz)(T_{b,z}f)(x)$ is also in L^2, for it is nothing but the L^2-multiplier,

$$\frac{d}{dz}(\widehat{T_{b,z}f})(\xi) = b\theta(\xi)(\log|\xi|)\frac{e^{i|\xi|^b}}{|\xi|^{bz-b/2}}\hat{f}(\xi).$$

This yields the boundedness of the first term. The boundedness of the second term follows by a straightforward differentiation as indicated. Likewise, when $\operatorname{Re} z = 1$ or $\operatorname{Re} z = 1/2$, it may be easily seen that $U(z)$ is a continuous function. We now verify the remaining hypotheses of Theorem (3.2).

If $\operatorname{Re} z = 1$, note that $T_{b,z}f = T_bR_yf$. Here R is the Hörmander multiplier given by $\widehat{R_yf}(\xi) = \hat{f}(\xi)/|\xi|^{iby}$, $f \in \hat{C}_{0,0}^\infty$. By using Theorem (3.4) on the operator R_y and Lemma (3.6) on T_b, we get for $\operatorname{Re} z = 1$,

$$\|T_{b,z}f\|_{1,v} \leqslant c(1+|y|^N)\|f\|_{H_v^1}.$$

By Plancherel's formula, if $\operatorname{Re} z = 1/2$, we get

$$\|T_{b,z}f\|_2 \leqslant c\|f\|_2, \qquad \operatorname{Re} z = 1/2.$$

Thus, by the conclusion of Theorem (3.2) in view of the two inequalities above we get

$$\|T_{b,1/p_0} f\|_{p_0,v^{2-p_0}} \leqslant c\|f\|_{p_0,v^{2-p_0}}.$$

But $T_{b,1/p_0}f = T_{b,a}f$, $a = b(1/p_0 - 1/2)$ and $w(x) = v^{2-p_0}$; the lemma follows. Q.E.D.

Having accomplished the interpolation aspects of Theorem C, we now turn to the geometric aspects of the proof of Theorem C. Before we proceed any further we wish to write down some properties of Whitney decompositions of open sets. All these properties are well known and their proofs may be found in Chapter VI of the book by Stein [**12**].

LEMMA (3.8). *Let Ω be any open set in R^n such that $R^n \setminus \Omega$ is nonempty. Then*:

(a) $\Omega = \bigcup I_j$, *I_j are cubes, whose interiors are pairwise disjoint. Moreover, the sides of the I_j's are parallel to the coordinate axes.*

(b) $\operatorname{diam} I_j \leqslant \operatorname{dist}(I_j, R^n \setminus \Omega) \leqslant 4 \operatorname{diam} I_j$.

(c) *Let $I_j^* = (9/8)I_j$; then* $\frac{3}{4}\operatorname{diam} I_j \leqslant \operatorname{dist}(I_j^*, R^n \setminus \Omega) \leqslant 4\operatorname{diam} I_j$.

(d) *Each point of Ω is contained in at most $(12)^n$ of the cubes I_j^*. This is the bounded overlap property.*

We are now ready to state the basic lemma for localizing the weights.

LEMMA (3.9). *Let Ω be any open set in R^n, such that $R^n \setminus \Omega$ is nonempty. Let $g(x) \geqslant 0$ be such that $g(x) = 0$ if $x \in R^n \setminus \Omega$ and $g^*(x) < \infty$ a.e. Then there exists a function $\rho(x) \geqslant 0$ such that $\rho^*(x) < \infty$ a.e. x with the properties that, for $\Omega = \bigcup I_j$ a Whitney decomposition satisfying properties* (a)–(d) *of the previous lemma, we have*:

(1) $c^{-1}g^*(x) \leqslant \rho^*(x) \leqslant cg^*(x)$ *a.e.* $x \in R^n \setminus \Omega$, *and* $c > 0$.

(2) $\sup_{x \in I_j^*} \rho^*(x) \leqslant c \operatorname{ess\,inf}_{I_j} g^*(x)$, $I_j^* = (9/8)I_j$.

PROOF. We define $\rho(x)$ by

$$\rho(x) = \begin{cases} \dfrac{1}{|I_j|}\displaystyle\int_{I_j} g(t)\,dt, & x \in I_j, \\ 0, & x \in R^n \setminus \Omega. \end{cases}$$

$\rho(x)$ is well defined as the cubes I_j are disjoint. Clearly $\rho(x) \geqslant 0$ and since $g(x) \in L^1_{\text{loc}}$, $\rho(x) \in L^1_{\text{loc}}$. Since $g^*(x) < \infty$ a.e., note that it follows from (1) and (2) that $\rho^*(x) < \infty$ a.e. We now show the left-hand inequality of (1). For $x \in R^n \setminus \Omega$, pick an interval I such that $x \in I$, and consider $|I|^{-1}\int_I g(t)\,dt$. Since $g(x)$ has support in Ω, consider those Whitney cubes I_k, such that $I \cap I_k \neq \varnothing$. The above average then may be estimated by

$$\tag{3.10} \begin{aligned} \frac{1}{|I|} \sum_{I \cap I_k \neq \varnothing} \int_{I_k} g(t)\,dt &= \frac{1}{|I|} \sum_{I \cap I_k \neq \varnothing} |I_k| \frac{1}{|I_k|} \int_{I_k} g(t)\,dt \\ &\leqslant \frac{1}{|I|} \sum_{I \cap I_k \neq \varnothing} \int_{I_k} \rho(t)\,dt. \end{aligned}$$

Now since $x \in R^n \setminus \Omega$, and $x \in I$, there exists a number $\alpha > 1$ dependent only on the dimension such that if $I \cap I_k \neq \varnothing$, then $I_k \subset \bar{I} = \alpha I$. This follows from (b) of Lemma (3.8). Hence the last term in the chain of estimates above may be estimated by

$$\frac{c}{|\bar{I}|} \sum_{I \cap I_k \neq \varnothing} \int_{I_k} \rho(t)\,dt \leqslant \frac{c}{|\bar{I}|} \int_{I_k} \rho(t)\,dt \leqslant \frac{c}{|\bar{I}|} \int_{\bar{I}} \rho(t)\,dt.$$

Hence it follows that since $x \in \bar{I}$, too, then for any $x \in R^n \setminus \Omega$, $g^*(x) \leqslant c\rho^*(x)$. The other half of the inequality (1) may be proved as follows. Fix I, $x \in I$, where $x \in R^n \setminus \Omega$. Then using the definition of $\rho(x)$ we get

$$\frac{1}{|I|} \int_I \rho(t)\,dt \leqslant \frac{1}{|I|} \sum_{I \cap I_k \neq \varnothing} \int_{I_k} \rho(t)\,dt.$$

But $\int_{I_k} \rho(t)\,dt = \int_{I_k} g(t)\,dt$. Hence the right side above is majorized by

$$\sum_{I \cap I_k \neq \varnothing} \frac{1}{|I|} \int_{I_k} g(t)\,dt \leqslant \frac{c}{|\bar{I}|} \int_{\cup I_k} g(t)\,dt \leqslant \frac{c}{|\bar{I}|} \int_{\bar{I}} g(t)\,dt.$$

Here $\bar{I} = \alpha I$ as before. The last term on the right side is bounded by $g^*(x)$. We have hence proved that $\rho^*(x) \leqslant cg^*(x)$ if $x \in R^n \setminus \Omega$. This yields (1).

We now show (2). For $x \in I_j^*$, fix I, $x \in I$. With no loss of generality we assume that I has its faces parallel to the axes. There are two natural cases which the estimates for (2) split into.

Case 1. $\operatorname{diam} I \leqslant \frac{1}{10} \operatorname{diam} I_j$. Consider now those Whitney cubes I_k, such that $I \cap I_k \neq \varnothing$. We now claim that there is a constant c dependent only on the dimension such that the number of I_k's is bounded above by c. Moreover, we claim that $\frac{1}{8} \operatorname{diam} I_j \leqslant \operatorname{diam} I_k \leqslant 10 \operatorname{diam} I_j$. The first claim really follows from the second one. For, by (a) of Lemma (3.8), if $I \cap I_k \neq \varnothing$ and I_k's are disjoint and both I_k and I have faces parallel to the axes, then indeed, if $\operatorname{diam} I_k \geqslant \frac{1}{8} \operatorname{diam} I_j \geqslant \frac{1}{80} \operatorname{diam} I$, the number of such I_k's has to be bounded by a constant c. So now we verify the second claim. Pick any x_0 in $R^n \setminus \Omega$. Now since $I \cap I_k \neq \varnothing$, $I \cap I_j^* \neq \varnothing$, and since $\operatorname{diam} I \leqslant \frac{1}{10} \operatorname{diam} I_j$, it follows that $\operatorname{dist}(I_k, I_j^*) \leqslant \frac{1}{4} \operatorname{diam} I_j$. But we have

$$\frac{3}{4} \operatorname{diam} I_j \leqslant \operatorname{dist}(x_0, I_j^*) \leqslant \operatorname{dist}(x_0, I_k) + \operatorname{dist}(I_k, I_j^*). \tag{3.11}$$

Substituting the estimate for $\operatorname{dist}(I_k, I_j^*)$ into (3.11) we get

$$\frac{1}{2} \operatorname{diam} I_j \leqslant \operatorname{dist}(x_0, I_k) = \operatorname{dist}(I_k, R^n \setminus \Omega).$$

The last equality follows because x_0 was arbitrary. By Lemma (3.8) we know that $\operatorname{dist}(I_k, R^n \setminus \Omega) \leqslant 4 \operatorname{diam} I_k$. Thus, substituting this above we see at once that $\frac{1}{8} \operatorname{diam} I_j \leqslant \operatorname{diam} I_k$. Hence we have verified one-half of the claim.

We prove the remaining part of the claim by contradiction. Assume $\operatorname{diam} I_j \leqslant \frac{1}{10} \operatorname{diam} I_k$. Since we know that $\operatorname{dist}(I_j^*, I_k) \leqslant \frac{1}{4} \operatorname{diam} I_j$, it follows that $\operatorname{dist}(I_j^*, I_k) \leqslant \frac{1}{40} \operatorname{diam} I_k$ and since $\operatorname{diam} I_k \leqslant \operatorname{dist}(x_0, I_k) \leqslant \operatorname{dist}(x_0, I_j^*) + \operatorname{dist}(I_j^*, I_k)$ it follows that

$$\operatorname{diam} I_k \leqslant \operatorname{dist}(x_0, I_j^*) + \frac{1}{40} \operatorname{diam} I_k.$$

Since x_0 is arbitrary the inequality above yields

$$\tfrac{39}{40}\operatorname{diam} I_k \leqslant \operatorname{dist}(x_0, I_j^*) \leqslant \operatorname{dist}(I_j^*, R^n \setminus \Omega) \leqslant 4 \operatorname{diam} I_j.$$

But since $\operatorname{diam} I_j \leqslant \frac{1}{10}\operatorname{diam} I_k$, the inequality above yields the contradiction, $\frac{39}{40}\operatorname{diam} I_k \leqslant \frac{2}{5}\operatorname{diam} I_k$. Thus, $\operatorname{diam} I_k \leqslant 10 \operatorname{diam} I_j$ as desired.

Now consider

$$\begin{aligned}\frac{1}{|I|}\int_I \rho(t)\,dt &= \frac{1}{|I|}\sum_{I\cap I_k \neq \varnothing}\int_{I\cap I_k}\rho(t)\,dt \\ &= \frac{1}{|I|}\sum_{I\cap I_k\neq\varnothing}|I\cap I_k|\frac{1}{|I_k|}\int_{I_k} g(t)\,dt.\end{aligned}$$

But from the claim the number of terms in the sum above is bounded by a fixed constant $c(n)$. So we can estimate further by

$$\max_{k=1,2,\ldots,c(n)}\frac{1}{|I_k|}\int_{I_k} g(t)\,dt\frac{1}{|I|}\sum_k |I\cap I_k| \leqslant \max_{k=1,2,\ldots,c(n)}\frac{1}{|I_k|}\int_{I_k} g(t)\,dt.$$

Using the fact that $\operatorname{dist}(I_k, I_j^*) \leqslant \frac{1}{4}\operatorname{diam} I_j$, and $\frac{1}{8}\operatorname{diam} I_j \leqslant \operatorname{diam} I_k \leqslant 10\operatorname{diam} I_j$, we can find a cube $\bar{I}$, such that $\bar{I} \supset I_k$, $k = 1,2,\ldots, c(n)$, $I_j \subset \bar{I}$ and $|\bar{I}|/|I_k| \leqslant c$ for all $k = 1,2,\ldots, c(n)$. Hence, we estimate the term above by

$$\frac{c}{|\bar{I}|}\int_{\bar{I}} g(t)\,dt \leqslant c \operatorname*{ess\,inf}_{I_j} g^*(x)$$

Case 2. $\operatorname{diam} I \geqslant \frac{1}{10}\operatorname{diam} I_j$. We wish to consider $|I|^{-1}\int_I \rho(t)\,dt$. It is evident that by simply expanding I, one may simply consider the case $\operatorname{diam} I \geqslant 10\operatorname{diam} I_j$. We claim that if $\{I_k\}$ is the subcollection of Whitney cubes such that $I \cap I_k \neq \varnothing$, then one can find a number $\alpha = \alpha(n)$ such that $\alpha I = \bar{I} \supset I_k$ for all k. To verify this it is enough to show that if $I \cap I_k \neq \varnothing$; then, $\operatorname{diam} I \geqslant \frac{1}{10}\operatorname{diam} I_k$. We proceed by contradiction. Assume $\operatorname{diam} I \leqslant \frac{1}{10}\operatorname{diam} I_k$. From these assumptions it follows that $\operatorname{diam} I_j \leqslant \frac{1}{100}\operatorname{diam} I_k$. Pick any $x_0 \in R^n \setminus \Omega$ such that $\operatorname{dist}(x_0, I_j^*) \leqslant 5\operatorname{diam} I_j \leqslant \frac{1}{20}\operatorname{diam} I_k$. But

$$\operatorname{dist}(x_0, I_k) \leqslant \operatorname{dist}(x_0, I_j^*) + \operatorname{dist}(I_j^*, I_k). \tag{3.12}$$

Now since $I \cap I_j^* \neq \varnothing$, and $I \cap I_k \neq \varnothing$, we have that $\operatorname{dist}(I_j^*, I_k) \leqslant \operatorname{diam} I \leqslant \frac{1}{10}\operatorname{diam} I_k$. So from (3.12) it follows that

$$\operatorname{dist}(x_0, I_k) \leqslant \tfrac{1}{20}\operatorname{diam} I_k + \tfrac{1}{10}\operatorname{diam} I_k \leqslant \tfrac{1}{5}\operatorname{diam} I_k.$$

But $\operatorname{diam} I_k \leqslant \operatorname{dist}(x_0, I_k)$. Thus we arrive at a contradiction. Hence, $\operatorname{diam} I \geqslant \frac{1}{10}\operatorname{diam} I_k$, and our claim is verified.

Letting $\operatorname{diam} I \geqslant 10\operatorname{diam} I_j$, we have

$$\begin{aligned}\frac{1}{|I|}\int_I \rho(t)\,dt &\leqslant \frac{1}{|I|}\sum_{I\cap I_k\neq\varnothing}\int_{I_k}\rho(t)\,dt \leqslant \frac{c}{|I|}\sum_{I\cap I_k\neq\varnothing}\int_{I_k} g(t)\,dt \\ &= \frac{c}{|I|}\int_{\cup I_k} g(t)\,dt \leqslant \frac{c}{|\bar{I}|}\int_{\bar{I}} g(t)\,dt.\end{aligned}$$

The last term on the right is bounded by $c\operatorname{ess\,inf}_{I_j} g^*(x)$. The lemma has been proved in entirety. Q.E.D.

4. The weak (1, 1) inequality. At this stage we are ready to prove Theorem C.

PROOF OF THEOREM C. We consider the open set $\Omega = \{t\colon f^*(t) > \lambda\}$. Since $\|f\|_{1,w} < \infty$, it is evident from (1.4) that $w(\Omega) \leqslant c\|f\|_{1,w}/\lambda < \infty$, because $w \in A_1$. Thus $R \setminus \Omega$ is nonempty. One can then obtain a Whitney decomposition for Ω from Lemma (3.8). Using (b) of Lemma (3.8) it follows that if $\Omega = \bigcup I_j$, then

$$\frac{1}{|I_j|}\int_{I_j}|f(t)|\,dt \leqslant c\lambda.$$

Moreover, if $x \in R \setminus \Omega$ then $|f(x)| \leqslant \lambda$. We write $f(x) = \psi(x) + \sum_j f_j(x)$, $\psi(x) = f(x)\chi_{R\setminus\Omega}(x)$ and $f_j(x) = f(x)\chi_{I_j}(x)$. We also let $h(x) = \sum_j f_j(x)$. We also assume that $f \in C_0^\infty$. If we prove Theorem C under this hypothesis, then the theorem follows by a routine density argument. Now

$$w\{x\colon |T_b f(x)| > \lambda\} \leqslant w\{x\colon |T_b\psi(x)| > \lambda/2\} + w\{x\colon |T_b h(x)| > \lambda/2\}. \tag{4.1}$$

By Chebychev's inequality the first term on the right side is bounded by

$$\frac{c}{\lambda^2}\int_R |T_b\psi|^2 w\,dx \leqslant \frac{c}{\lambda^2}\int_R |\psi|^2 w\,dx \leqslant \frac{c}{\lambda}\|f\|_{1,w}.$$

Note that for the first inequality we used Theorem A and the fact that if $w \in A_1$ then $w \in A_2$. For the second inequality, we used the definition of $\psi(x)$. Hence we are left to estimate the second term on the right side in (4.1).

Letting $\tilde{\Omega} = \bigcup 4I_j$, we have

$$w\{x\colon |T_b h(x)| > \lambda/2\} \leqslant w\{x\colon R \setminus \tilde{\Omega}, |T_b h(x)| > \lambda/2\} + w(\tilde{\Omega}).$$

But since w is doubling, we have

$$w(\tilde{\Omega}) \leqslant \sum_j w(4I_j) \leqslant c\sum_j w(I_j) \leqslant cw(\Omega) \leqslant \frac{c}{\lambda}\|f\|_{1,w}.$$

Now choose a function $\phi \in C_0^\infty(R)$ and $\phi \geqslant 0$ such that $\int\phi(x)\,dx = 1$ and $\phi(x)$ is supported in $(-1/2, 1/2)$. We let $\delta_j = |I_j|$, and let $\phi_j(x) = \delta_j^{-1/(1-b)}\phi(x/\delta_j^{1/(1-b)})$. We eliminate the trivial terms very much like [**6**]. Let

$$h(x) = \sum_{\substack{j\\ |I_j| \leqslant \delta_b}} f_j(x) + \sum_{\substack{j\\ |I_j| > \delta_b}} f_j(x) = h_1(x) + h_2(x).$$

Here δ_b is such that $\delta_b = [1/100]^{(1-b)/b}$. By (1.1) we get

$$w\{x\colon R \setminus \tilde{\Omega}, |T_b h_2(x)| > \lambda/2\} \leqslant \frac{c}{\lambda}\sum_j \iint_{\substack{I_j\\ |x-x_j| \geqslant 2|I_j|}} |f(t)|\frac{w(x)}{|x - x_j|^2}\,dt\,dx, \tag{4.2}$$

x_j = center of I_j.

Since $|I_j| > \delta_b$ and $w \in A_1$, a standard argument yields

$$\int_{|x-x_j| \geqslant 2|I_j|}\frac{w(x)\,dx}{|x - x_j|^2} \leqslant c\,\operatorname*{ess\,inf}_{I_j} w(x).$$

Substituting this in the right side of (4.2) we easily see that

$$w\{x\colon R \setminus \tilde{\Omega}, |T_b h_2(x)| > \lambda/2\} \leqslant \frac{c}{\lambda}\|f\|_{1,w}.$$

We will show that

$$w\{x: R\setminus\tilde{\Omega}, |T_b h_1(x)| > \lambda/2\} \leqslant \frac{c}{\lambda}\|f\|_{1,w}. \tag{4.3}$$

Now, consider the difference,

$$T_b(h_1)(x) - T_b\Bigg(\sum_{\substack{j\\|I_j|\leqslant\delta_b}} \phi_j * f_j\Bigg)(x), \qquad x\in R\setminus\tilde{\Omega}.$$

Using (1.1) we may estimate the term above as follows:

$$|T_b(h_1)(x) - T_b\Big(\sum_j \phi_j * f_j\Big)(x)| \leqslant \sum_j |T_b(f_j)(x) - T_b(\phi_j * f_j)(x)|.$$

Note that one is allowed the interchange of the summation and T_b since $f_j(x)$ comes from $f(x)$ which is smooth. Now for $x\in R\setminus\tilde{\Omega}$ and $b' = b/(1-b)$,

$$\begin{aligned}|T_b(f_j)(x) &- T_b(\phi_j * f_j)(x)|\\ &\leqslant c\iint_{I_j}\left|\frac{e^{i\alpha_b|x-y-t|^{-b'}}}{|x-y-t|} - \frac{e^{i\alpha_b|x-y|^{-b'}}}{|x-y|}\right||f_j(y)|\phi_j(t)\,dy\,dt\\ &\quad + c\left(\int_{I_j}|f_j(y)|dy\right)\left(\frac{1}{(1+|x-x_j|)^2} + \frac{\chi(|x-x_j|\leqslant c)}{|x-x_j|^{1-\varepsilon}}\right).\end{aligned}$$

Using the mean value theorem for the first term we see that for x not in $4I_j$,

$$\begin{aligned}|T_b(f_j)(x) &- T_b(\phi_j * f_j)(x)|\\ &\leqslant c\left(\int_{I_j}|f_j(y)|dy\right)\left[\frac{\delta_j^{1/(1-b)}}{(\delta_j+|x-x_j|)^{(b'+2)}}\right.\\ &\qquad\qquad\left. + \frac{\chi(|x-x_j|\leqslant c)}{|x-x_j|^{1-\varepsilon}} + \frac{1}{(1+|x-x_j|)^2}\right].\end{aligned} \tag{4.4}$$

Thus, from (4.4) it follows that

$$\begin{aligned}\sum_j \int_{R\setminus\tilde{\Omega}} &|T_b(f_j) - T_b(\phi_j * f_j)|w\,dx\\ &\leqslant c\sum_j\left(\int_{I_j}|f_j|dy\right)\int_{R\setminus 4I_j}\left[\frac{\delta_j^{1/(1-b)}}{(\delta_j+|x-x_j|)^{(2-b)/(1-b)}}\right.\\ &\qquad\qquad\left. + \frac{\chi(|x-x_j|\leqslant c)}{|x-x_j|^{1-\varepsilon}} + \frac{1}{(1+|x-x_j|)^2}\right]w\,dx.\end{aligned} \tag{4.5}$$

But since w is in A_1, the second integral on the right is easily seen to be bounded by $c\operatorname{ess\,inf}_{I_j} w$. Thus the right side of (4.5) is bounded by

$$c\sum_j\Big(\operatorname*{ess\,inf}_{I_j} w\Big)\int_{I_j}|f_j|dy \leqslant c\|f\|_{1,w}.$$

Thus the term on the left in (4.3) is bounded by

$$\frac{c}{\lambda}\|f\|_{1,w} + w\left\{x \in R\setminus\tilde{\Omega} : \left|T_b\left(\sum_j \phi_j * f_j\right)(x)\right| > \lambda/4\right\},$$

the sum over j being of course taken over those I_j for which $|I_j| \leqslant \delta_b$.

It is at this stage the estimates become delicate. Using Chebychev's inequality,

$$(4.6)\quad w\left\{x \in R\setminus\tilde{\Omega} : \left|T_b\left(\sum_j \phi_j * f_j\right)(x)\right| > \lambda/4\right\} \leqslant \frac{c}{\lambda^{p_0}}\int_{R\setminus\tilde{\Omega}}\left|T_b\left(\sum_j \phi_j * f_j\right)\right|^{p_0} w\,dx.$$

The number p_0 is the same for which Lemma (3.5) holds. Now by Theorem (3.1), $w(x) \leqslant c(g^*(x))^\sigma$, for some σ, $0 < \sigma < 1$. Let $g(x) = g(x)\chi_\Omega(x) + g(x)\chi_{R\setminus\Omega}(x) = g_1(x) + g_2(x)$. Hence, $w(x) \leqslant c((g_1^*(x))^\sigma + (g_2^*(x))^\sigma)$. The function $g_1(x)$ satisfies all the hypotheses of Lemma (3.9), hence there is a function $\rho(x)$, such that for $x \in R\setminus\tilde{\Omega}$, $(g_1^*(x))^\sigma \leqslant c(\rho^*(x))^\sigma$. Note that by Theorem (3.1), $(\rho^*(x))^\sigma$ is a weight in A_1. We call this weight $w_1(x)$. We denote $(g_2^*(x))^\sigma$ as $w_2(x)$. By Lemma (3.9) and Theorem (3.1) we have $\sup_{x\in I_j^*} w_1(x) \leqslant c\operatorname{ess\,inf}_{I_j} w(x)$, $I_j^* = \frac{9}{8}I_j$. Note, too, that $w_2(x)$ is also a weight in A_1 and $w_2(x) \leqslant cw(x)$ for all x. So by Lemma (3.9),

$$(4.7)\quad \frac{c}{\lambda^{p_0}}\int_{R\setminus\tilde{\Omega}}\left|T_b\left(\sum_j \phi_j * f_j\right)\right|^{p_0} w\,dx \leqslant \frac{c}{\lambda^{p_0}}\int_R\left|T_b\left(\sum_j \phi_j * f_j\right)\right|^{p_0}(w_1 + w_2)\,dx.$$

By construction of $w_1(x)$ and $w_2(x)$ (the σ being the same for both), Lemma (3.5) applies with p_0 being the same for both $w_1(x)$ and $w_2(x)$. We now claim the inequality

$$(4.8)\quad \frac{1}{\lambda^{p_0}}\int_R\left|T_b\left(\sum_j \phi_j * f_j\right)\right|^{p_0}(w_1 + w_2)\,dx$$

$$\leqslant \frac{c}{\lambda^{p_0}}\int_R\left|G_{b/p_0'}\left(\sum_j \phi_j * f_j\right)\right|^{p_0}(w_1 + w_2)\,dx,$$

$1/p_0 + 1/p_0' = 1$. The operator $G_{b/p_0'}$ is the multiplier.

$$\widehat{G_{b/p_0'}f}(\xi) = \frac{\eta(\xi)}{|\xi|^{b/p_0'}}\hat{f}(\xi).$$

$\eta(\xi)$ is a C^∞ cut-off function, such that

$$\eta(\xi) = \begin{cases} 1, & |\xi| \geqslant 1/4, \\ 0, & |\xi| \leqslant 1/8. \end{cases}$$

To see (4.8), we observe that T_b comes in two pieces as follows:

$$\widehat{T_b f}(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^{b/2 - b/p_0'}}\frac{\eta(\xi)}{|\xi|^{b/p_0'}}\hat{f}(\xi).$$

Hence,

$$T_b f(x) = T_{b,a}(G_{b/p_0'} f)(x), \qquad a = b(1/2 - 1/p_0').$$

Applying Lemma (3.5) to the operator $T_{b,a}$ immediately yields (4.8). We now let $\bar{I}_j = \frac{17}{16} I_j$. Note $I_j \subset \bar{I}_j \subset I_j^*$, $I_j^* = \frac{9}{8} I_j$.

Our aim is to show that

$$\int_R \left| G_{b/p_0'}\left(\sum_j \phi_j * f_j \right) \right|^{p_0} w_1 \, dx \leqslant c\lambda^{p_0 - 1} \|f\|_{1,w}, \tag{4.9}$$

$$\int_R \left| G_{b/p_0'}\left(\sum_j \phi_j * f_j \right) \right|^{p_0} w_2 \, dx \leqslant c\lambda^{p_0 - 1} \|f\|_{1,w}. \tag{4.10}$$

If the two estimates above are proved then it follows from (4.8), (4.7) and (4.6) that

$$w\left\{ x \colon R \setminus \tilde{\Omega} \colon \left| T_b\left(\sum_j \phi_j * f_j \right)(x) \right| > \lambda \right\} \leqslant \frac{c}{\lambda} \|f\|_{1,w}.$$

This is the desired estimate. Now

$$\begin{aligned} G_{b/p_0'}\left(\sum_j \phi_j * f_j \right)(x) &= \sum_j G_{b/p_0'}(\phi_j * f_j)(x) \chi_{\bar{I}_j}(x) \\ &\quad + \sum_j G_{b/p_0'}(\phi_j * f_j)(x) \chi_{\bar{I}_j^c}(x) \\ &= A + B. \end{aligned} \tag{4.11}$$

It is well known that the kernel associated with $G_{b/p_0'}$, which we will also call $G_{b/p_0'}$, satisfies the estimates

$$|G_{b/p_0'}(x)| \leqslant \frac{c\chi(|x| \leqslant 1)}{|x|^{1 - b/p_0'}} + \frac{c\chi(|x| \geqslant 1)}{|x|^2} = \Phi(x). \tag{4.12}$$

So using (4.12) in B we get

$$\begin{aligned} |B| &\leqslant \sum \int_{I_j} \Phi * \phi_j(x - y) |f_j(y)| \, dy \leqslant c \sum_j \sup_{y \in I_j} |\Phi * \phi_j(x - y)| \int_{I_j} |f_j| \, dy \\ &\leqslant c \sum_j \sup_{y \in I_j} \Phi * \phi_j(x - y) |I_j| \left(\frac{1}{|I_j|} \int_{I_j} |f_j| \, dy \right) \\ &\leqslant c\lambda \sum_j \int_{I_j} \Phi * \phi_j(x - y) \, dy \leqslant c\lambda \Phi * \sum_j (\phi_j * \chi_{I_j})(x). \end{aligned}$$

But $\Phi \in L^1$ and $\|\Sigma_j(\phi_j * \chi_{I_j})(x)\|_\infty \leqslant c$ because of bounded overlaps. Hence, $\|\Phi * \Sigma_j(\phi_j * \chi_{I_j})\|_\infty \leqslant c$. So we get that $|B| \leqslant \lambda$. Thus,

$$\int_R |B|^{p_0} (w_1(x) + w_2(x)) \, dx \leqslant c\lambda^{p_0 - 1} \int_R |B| (w_1 + w_2) \, dx. \tag{4.13}$$

Now we simply consider $\int_R |B| w_1\, dx$. The term that remains may be estimated similarly:

$$\int_R |B| w_1\, dx \leqslant c \sum_j \int_{\bar{I}_j^c} \left(\int_{I_j} \Phi(x-y) |\phi_j * f_j(y)|\, dy \right) w_1\, dx$$

$$\leqslant c \sum_j \left(\int_{\bar{I}_j^c} \Phi(x - x_j) w_1\, dx \right) \int_{I_j} |f_j|\, dy,$$

since $\|\phi_j\|_1 = 1$. Using (4.12) and a standard argument easily yields that, for $w_1 \in A_1$,

$$\int_{\bar{I}_j^c} \Phi(x - x_j) w_1(x)\, dx \leqslant c \operatorname*{ess\,inf}_{\bar{I}_j} w_1(x) \leqslant c \operatorname*{ess\,inf}_{I_j} w.$$

The last estimate is valid because of Lemma (3.9). In case we were estimating the part with $w_2(x)$, in the last inequality above, we would have used the obvious estimate, $c \operatorname{ess\,inf}_{\bar{I}_j} w_2(x) \leqslant c \operatorname{ess\,inf}_{\bar{I}_j} w(x)$. So the right side of (4.13) is majorized by $c\lambda^{p_0 - 1} \|f\|_{1,w}$. So we need to show that

$$\int_R |A|^{p_0} (w_1 + w_2)\, dx \leqslant c\lambda^{p_0 - 1} \|f\|_{1,w}.$$

Now, by bounded overlaps,

$$\int_R |A|^{p_0} w_1\, dx \leqslant c \sum_j \int_{\bar{I}_j} |G_{b/p_0'} * \phi_j * f_j|^{p_0} w_1(x)\, dx.$$

Using Lemma (3.9) we see the right side is majorized by

$$(4.14) \quad c \sum_j \sup_{\bar{I}_j} w_1(x) \int_{\bar{I}_j} |G_{b/p_0'} * \phi_j * f_j|^{p_0}\, dx$$

$$\leqslant c \sum_j \left(\operatorname*{ess\,inf}_{I_j} w(x) \right) \int_R |G_{b/p_0'} * \phi_j * f_j|^{p_0}\, dx$$

$$\leqslant c \sum_j \left(\operatorname*{ess\,inf}_{I_j} w(x) \right) \int_R |G_{b/p_0'} * \phi_j|^{p_0}\, dx \left(\int_{I_j} |f_j|\, dx \right)^{p_0}.$$

But

$$\left(\int_{I_j} |f_j|\, dx \right)^{p_0} \leqslant c\lambda^{p_0 - 1} |I_j|^{p_0 - 1} \int_{I_j} |f|\, dx.$$

Moreover, by the Hardy-Littlewood-Sobolev fractional integration theorem,

$$\int_R |G_{b/p_0'} * \phi_j|^{p_0}\, dx \leqslant c |I_j|^{1 - p_0}.$$

Hence, (4.14) is majorized by

$$\leqslant c\lambda^{p_0 - 1} \sum_j \left(\operatorname*{ess\,inf}_{I_j} w \right) \int_{I_j} |f|\, dx \leqslant c\lambda^{p_0 - 1} \|f\|_{1,w}.$$

Now,

$$\int_R |A|^{p_0} w_2(x)\,dx \leqslant c\sum_j \int_{\bar{I}_j} |G_{b/p_0'} * \phi_j * f_j|^{p_0} w_2(x)\,dx. \tag{4.15}$$

But we recall that $w_2(x) = [(g(x)\chi_{R\setminus\Omega})^*(x)]^\sigma$. We also recall from Lemma (3.8) that $\Omega = \bigcup I_j^*$, $I_j^* = \frac{9}{8}I_j$. The cube $\bar{I}_j = \frac{17}{16}I_j$. So evidently by the geometry, for $x \in \bar{I}_j$, $w_2(x) \leqslant c\sup_I(|I|^{-1}\int_I g(t)\,dt)^\sigma \leqslant c\operatorname{ess\,inf}_{I_j} w(x)$. The supremum being taken only over those I, such that $I_j \subset I$.

Thus (4.15) is majorized by

$$c\sum_j \Big(\operatorname*{ess\,inf}_{I_j} w(x)\Big)\int_R |G_{b/p_0'} * \phi_j * f_j|^{p_0}\,dx.$$

Hence, proceeding as in (4.14) the term above is bounded by $c\lambda^{p_0-1}\|f\|_{1,w}$ as before. This establishes the estimates (4.9) and (4.10) and hence Theorem C. Q.E.D.

The following corollary is evident from Theorem (1.4), Theorem C and Lemma (1.2).

COROLLARY (4.16). *Consider for $x \in R$,*

$$K_b(x) = \frac{e^{i|x|^{-b}}}{|x|}\chi(|x| \leqslant 1), \qquad 0 < b < \infty.$$

If $w \in A_1$, then $w\{x\colon |K_b * f(x)| > \lambda\} \leqslant (c/\lambda)\|f\|_{1,w}$, $\lambda > 0$.

5. Inequalities for $T_{b,a}$ and Theorem D.

PROOF OF THEOREM B. The proof is fairly standard. We employ Theorem (5.5.3) of [**1**]. To do so we consider the analytic family of operators,

$$T_{b,z}f(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^z}\hat{f}(\xi), \qquad z = u + iy.$$

By the results of Fefferman and Stein in [**7**] (see (0.2)), it follows that, if $\operatorname{Re} z = nb|1/p - 1/2|$, then

$$\|T_{b,z}f\|_p \leqslant c_p(1 + |y|)^N\|f\|_p. \tag{5.1}$$

But if $\operatorname{Re} z = nb/2$, Theorem A yields for $w \in A_p$,

$$\|T_{b,z}f\|_{p,w} \leqslant c_p(1 + |y|)^N\|f\|_{p,w}. \tag{5.2}$$

Interpolating between (5.1) and (5.2) we have that, if $\gamma = (a - \alpha)/(nb/2 - \alpha)$, then

$$\|T_{b,a}f\|_{p,w^\gamma} \leqslant c_p\|f\|_{p,w^\gamma}. \tag{5.3}$$

Thus we get Theorem B. Q.E.D.

REMARK. It is easy to see that for the operators T_b and weights $w(x) = (1 + |x|)^\alpha$, $\alpha \in R$, one has for $1 < p < \infty$,

$$\|T_bf\|_{p,w} \leqslant c\|f\|_{p,w}.$$

It is enough to see this for $p = 2$, and α an even integer. The remaining cases follow trivially from interpolation with the unweighted case and hence by duality. Now if $\alpha = 4k$,

$$\int_{R^n} |T_b f|^2 |x|^{4k} dx \leqslant \sum_{j=1}^{n} \int_{R^n} \left| \frac{\partial^{2k}}{\partial \xi_j^{2k}} \left(\theta(\xi) \frac{e^{i|\xi|^b}}{|\xi|^{nb/2}} \hat{f}(\xi) \right) \right|^2 d\xi.$$

Straightforward differentiation by Leibniz's formula yields that the right side is bounded by $\int_{R^n} |f|^2 (1 + |x|)^{4k} dx$. Hence, it follows that for $w(x) = (1 + |x|)^{4k}$, $k = 0, 1, 2 \ldots,$

$$\|T_b f\|_{2,w} \leqslant c\|f\|_{2,w}.$$

The remark follows.

THEOREM (5.4). *Let $w(x) = |x|^\alpha$. If $\alpha \leqslant -n$ or $\alpha \geqslant n(p-1)$ then the inequality $\|T_b f\|_{p,w} \leqslant c\|f\|_{p,w}$ is false.*

PROOF. The proof is by contradiction. We give a proof for $n = 1$. Note that by duality it is enough to show that when $\alpha = p - 1$, the inequality is false. Having made this reduction we proceed with the proof.

Let $\phi(\xi) \in C_0^\infty(R)$ such that

$$\phi(\xi) = \begin{cases} 1, & |\xi| \leqslant 2, \\ 0, & |\xi| \geqslant 4. \end{cases}$$

We let $\phi_N(\xi) = \phi(\xi/N)$, $N \geqslant 4$. Consider the function $\hat{f}_N(\xi) = \phi_N(\xi)$. Trivially, we see that

(5.5) $\qquad f_N(x) = Nf(Nx)$, where $f(x)$ is a Schwartz function.

Thus from (5.5) and changing variables we get

$$\int_R |f_N|^p |x|^{p-1} dx \leqslant c. \tag{5.6}$$

Our next aim is to estimate $T_b f_N(x)$ from below. In particular, we will show that, for $x \leqslant 0$ and $2bN^{b-1} \leqslant 2^{b-1}$, $0 < b < 1$,

$$|T_b f_N(x)| \geqslant c/|x|. \tag{5.7}$$

If (5.7) is established then the lemma follows. For if (5.7) were true, then, if $x \leqslant 0$, $N^{b-1} \leqslant |x| \leqslant 2^{b-1}$,

$$\int_R |T_b f_N|^p |x|^{p-1} dx \geqslant c \int \frac{dx}{|x|} \geqslant c \log N \quad \text{for large } N. \tag{5.8}$$

But if $\|T_b f\|_{p,|x|^{p-1}} \leqslant c_p \|f\|_{p,|x|^{p-1}}$ were true, it follows from (5.6) and (5.8) that $\log N \leqslant c$. By letting $N \to \infty$ we get a contradiction.

We now verify (5.7). Clearly,

$$T_b f_N(x) = \int_R \theta(\xi) \phi_N(\xi) \frac{e^{i(|\xi|^b + \xi x)}}{|\xi|^{b/2}} d\xi.$$

We now simply do a stationary phase computation. First we change variables in the integral above by setting $\xi = t|x|^{1/(b-1)}$. Using $x \leqslant 0$ we get

$$T_b f_N(x) = |x|^{(1-b/2)/(b-1)} \int_R \phi_N\big(t|x|^{1/(b-1)}\big)\theta\big(t|x|^{1/(b-1)}\big)\frac{e^{i|x|^{b/(b-1)}(|t|^b - t)}}{|t|^{b/2}}\,dt.$$

We split the integral above into two pieces as follows:

$$= |x|^{(1-b/2)/(b-1)} \int_{(b/2)^{1/(1-b)}}^{(2b)^{1/(1-b)}} \phi_N\big(t|x|^{1/(b-1)}\big)\theta\big(t|x|^{1/(b-1)}\big)\frac{e^{i|x|^{b/(b-1)}(t^b - t)}}{|t|^{b/2}}\,dt$$

$$+\,\text{remainder}.$$

We note that for the remainder the phase function $t^b - t$ does not possess a stationary point. Thus the remainder may be treated by an integration by parts.

We note that the remainder is given by

$$|x|^{(1-b/2)/(b-1)}\left(\int_{|t|\leqslant(b/2)^{1/(1-b)}} + \int_{(2b)^{1/(1-b)}\leqslant|t|\leqslant N|x|^{-1/(b-1)}}\right).$$

Recall too that $|x| \geqslant 2bN^{(b-1)}$, $0 < b < 1$, hence the last term above in brackets does not appear. Consider, then, just the part $t \geqslant 0$ of the first integral above to get

$$|x|^{(1-b/2)/(b-1)} \int_0^{(b/2)^{1/(1-b)}} \phi_N\big(t|x|^{1/(b-1)}\big)\theta\big(t|x|^{1/(b-1)}\big)\frac{e^{i|x|^{b/(b-1)}(t^b - t)}}{|t|^{b/2}}\,dt.$$

Integration by parts gives us

$$\left.\frac{|x|^{(1-b/2)/(b-1)}e^{i|x|^{b/(b-1)}(t^b-t)}\phi_N\big(t|x|^{1/(b-1)}\big)\theta\big(t|x|^{1/(b-1)}\big)}{(bt^{b-1}-1)|x|^{b/(b-1)}t^{b/2}}\right]_0^{(b/2)^{1/(1-b)}}$$

$$+c\int_0^{(b/2)^{1/(1-b)}} |x|^{(1-3b/2)/(b-1)}\frac{e^{i|x|^{b/(b-1)}(t^b-t)}}{(bt^{b-1}-1)t^{b/2}}$$

$$\times\left[\phi_N'\big(t|x|^{1/(b-1)}\big)\theta\big(t|x|^{1/(b-1)}\big) + \phi\theta_N' + \frac{\phi_N\theta}{t}\right]dt.$$

But if $0 < t \leqslant (b/2)^{1/(1-b)}$, we see that, $|bt^{b-1} - 1| \geqslant ct^{b-1}$, thus for the terms above we get, keeping the supports of θ and ϕ_N in mind,

$$c|x|^{(1-3b/2)/(b-1)}\left(1 + \int_0^{(b/2)^{1/(1-b)}} t^{1-3b/2}\left(|\phi_N'||\theta| + |\phi_N||\theta'| + \frac{|\phi_N||\theta|}{t}\right)dt\right).$$

Using $2bN^{b-1} \leqslant |x| \leqslant 1$ and performing the integrations above, we easily see the remainder is $O(1/|x|^{1-b/2(1-b)})$.

For the dominant term above, if $|x|^{1/(b-1)} \leqslant N$, from the definitions, $\phi_N(t|x|^{1/(b-1)}) \equiv 1$. Likewise, if $|x|^{1/(b-1)} \geqslant 2$ then $\theta(t|x|^{1/(b-1)}) \equiv 1$. Thus for $N^{b-1} \leqslant |x| \leqslant 2^{b-1}$, the integral equals

$$|x|^{(1-b/2)/(b-1)} \int_{(b/2)^{1/(1-b)}}^{(2b)^{1/(1-b)}} e^{i|x|^{b/(b-1)}(t^b-t)}|t|^{-b/2}\,dt. \tag{5.9}$$

Using the stationary phase principle (for example, [**4**, p. 31]), we see immediately that as $|x| \to 0$ the integral above is equal to

$$\frac{ce^{i\alpha_b|x|^{b/(b-1)}}}{|x|} + O\left(\frac{1}{|x|^{1-b/2(1-b)}}\right), \qquad 0 < b < 1, \alpha_b = b^{b/(1-b)} - b^{1/(1-b)}.$$

The remainder too is $O(1/|x|^{1-b/2(1-b)})$, $0 < b < 1$. Since $|x| \to 0$, or what is in other words $N \to \infty$, (5.7) is now evident. Q.E.D.

REMARKS. 1. For the case when $n > 1$, we define $f_N(x)$ in an analogous manner. However, we need to show here that the estimate corresponding to (5.7) is

$$|T_b f_N(x)| \geqslant c/|x|^n \quad \text{if } N^{b-1} \leqslant |x| \leqslant 2^{b-1}.$$

Using the Hankel transform formula from [**13**],

$$(5.10) \quad T_b f_N(x) = c|x|^{-(n-2)/2} \int_0^\infty \theta(r)\phi_N(r) r^{n(1-b)/2} e^{ir^b} J_{(n-2)/2}(r|x|)\, dr.$$

$J_{(n-2)/2}(r)$ is the Bessel function of order $(n-2)/2$. $\theta(r)$ is the function $\theta(|x|)$ and likewise $\phi_N(r)$. We now use the asymptotics of the Bessel function (see, for example, [**17**, p. 42]):

$$J_{(n-2)/2}(r) = (e^{ir} + e^{-ir}) \sum_{j=0}^{m} \alpha_j r^{-(j+1/2)} + O(r^{-m-3/2}).$$

Choose m so that $m + 3/2 > n(1-b)/2$. The error term in (5.10) is thus $O(|x|^{-(n-2)/2})$. By integration by parts one may readily see that the dominant term in (5.10) is

$$c|x|^{-(n-1)/2} \int_0^\infty \theta(r)\phi_N(r) r^{n(1-b)/2-1/2} e^{i(r^b - r|x|)}\, dr.$$

This term arises if, in (5.10), we substitute just the first term of the asymptotic expansion for $J_{(n-2)/2}(r)$. Note that we ignore the terms in e^{ir} for the simple reason that they do not have a stationary point in the phase, and thus integration by parts can easily handle them. In the integral above, we change variables by setting $r = |x|^{1/(b-1)} t$ to get

$$c|x|^{-n+b/2(b-1)} \int_0^\infty \theta\left(t|x|^{1/(b-1)}\right)\phi_N\left(t|x|^{1/(b-1)}\right) t^{n(1-b)/2-1/2} e^{i|x|^{b/(b-1)}(t^b - t)}\, dt.$$

We break up the integral into two pieces by splitting up the integral for $(b/2)^{1/(1-b)} \leqslant t \leqslant (2b)^{1/(1-b)}$ and the remainder. This is now exactly the case for $n = 1$. The remainder is handled by integration by parts as usual. Applying the principle of stationary phase to the dominant term we get

$$= \frac{ce^{i\alpha_b|x|^{b/(b-1)}}}{|x|^n} + O\left(|x|^{-n+b/2(1-b)}\right).$$

α_b, as when $n = 1$, is equal to $b^{b/(1-b)} - b^{1/(1-b)}$.

2. The techniques used above may also be used to show that in the context of A_p weights, Theorem B is sharp.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903

Current address: Department of Mathematics, Indiana University, Bloomington, Indiana 47405